ACCELERATION OF AN UNBALANCED ROTOR THROUGH ITS CRITICAL SPEEDS

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Submitted in partial fulfillment of
the requirements for the degree of

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IN
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ABSTRACT

The motion of an unbalanced rotor during acceleration through its critical speed is studied by numerical solutions obtained with a digital computer. The rotor is laterally restrained in two orthogonal directions by linear springs and is accelerated by a constant applied torque. It is found that, for a fixed combination of rotor unbalance and lateral stiffnesses, the applied torque must exceed a limiting value in order to accelerate successfully through the critical speed region. With smaller applied torques, after initial acceleration from rest the speed oscillates continually about the critical speed and the lateral excursions grow steadily. The conditions necessary for a successful acceleration are established and the maximum lateral excursions during successful accelerations are determined. The effects of small amounts of viscous damping in the lateral directions are also obtained. Finally, an example problem is solved to illustrate the extension of the solution to those problems where applied torque is a function of speed.

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NOTATION

c_1 = Damping in x direction

c_{1c} = Critical damping in x direction

c_2 = Damping in y direction

c_{2c} = Critical damping in y direction

e = Eccentricity of mass center from center of rotation

I = Moment of inertia about a longitudinal axis through the center of rotation

I_o = Moment of inertia about a longitudinal axis through the mass center

k_1 = Combined shaft and bearing stiffness in the x direction

k_2 = Combined shaft and bearing stiffness in the y direction

lbf = Pounds force

lbm = Pounds mass

m = Mass of the system

M = Applied torque

P(1) = Eccentricity parameter me^2/I

P(2) = Torque parameter Mm/Ik_1

P(3) = Stiffness ratio k_2/k_1

P(4) = Damping parameter $(2c_1/c_{1c}) = (2c_2/c_{2c})$

P(5) = Modifying parameter for applied torque

R/e = Dimensionless resultant deflection

ζ = Dimensionless deflection in the x direction

η = Dimensionless deflection in the y direction

θ = Angle measured counter clockwise from x axis to radial axis originating at the center of rotation and passing through the mass center

ξ = Dimensionless time

x, y = Coordinates of center of rotation

x_o, y_o = Coordinates of mass center

1. Introduction.

The maximum amplitude of vibration of an unbalanced rotor upon acceleration through its critical speed is of real importance to design engineers.

An analytic solution was obtained by Lewis [1]* for the linear single degree of freedom system having constant acceleration and constant force amplitude, with and without damping. Baker [2] obtained a mathematical machine solution of a linear two degree of freedom system with constant acceleration, also with and without damping. Meuser and Weibel [3] obtained a solution on the mechanical analyzer for a single degree of freedom system having constant acceleration and linear plus cubic elasticity, with and without damping. An analog computer solution to the damped and undamped linear two degree of freedom system was obtained by McCann and Bennett [4] . Here again a constant acceleration was assumed. Dornig [5] solved analytically the undamped single degree of freedom system with constant acceleration.

Both Biezeno and Grammel [6] and Baker [2] mention the possibility of an unsuccessful acceleration; that is, an acceleration to the vicinity of but not through the critical speed. In this case, the energy supplied is absorbed by the vibrations and damping, if present, rather than being absorbed by the rotor itself in the form of kinetic energy of rotation.

In the material that follows, a system having two lateral degrees of freedom is investigated. The rotor is accelerated by a constant applied torque rather than having a constant angular acceleration. For this system, the maximum amplitudes of vibration will be obtained for an appropriate range of dimensionless parameters. Included will be the damped and undamped cases, and those of equal and unequal spring constants in the x and y directions. In addition, the areas of successful

* Square brackets refer to bibliography

and unsuccessful acceleration will be defined. Finally, an example will illustrate the extension of the solution to those problems where torque is a function of speed.

2. Equations of Motion.

Figure 1* shows the idealized system. The rotating mass m has moment of inertia I_o about the longitudinal axis through the mass center which is displaced a distance e from the center of the supporting shaft S. Lateral displacement is resisted by the stiffnesses of the shaft and its bearings, k_1 and k_2, and the associated viscous damping c_1 and c_2. A driving torque M accelerates the rotor.

Figure 2 shows the free body diagram of the system and includes all real and D'Alembert forces. The origin of the coordinate system is at point B, which is the location of the shaft center S when the rotor is undeflected.

Summing forces and moments:

$$\Sigma F_x: \quad m\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} + k_1 x = me\left(\frac{d\theta}{dt}\right)^2 \cos\theta + me\left(\frac{d^2\theta}{dt^2}\right)\sin\theta \tag{1}$$

$$\Sigma F_y: \quad m\frac{d^2y}{dt^2} + c_2\frac{dy}{dt} + k_2 y = me\left(\frac{d\theta}{dt}\right)^2 \sin\theta - me\left(\frac{d^2\theta}{dt^2}\right)\cos\theta \tag{2}$$

$$\Sigma M_S: \quad I_o\frac{d^2\theta}{dt^2} = M - me^2\left(\frac{d^2\theta}{dt^2}\right) - me\left(\frac{d^2y}{dt^2}\right)\cos\theta + me\left(\frac{d^2x}{dt^2}\right)\sin\theta \tag{3}$$

Taking the moment of inertia I about the shaft center S rather than the mass center reduces Eq. 3 to:

$$I\frac{d^2\theta}{dt^2} = M - me\left(\frac{d^2y}{dt^2}\right)\cos\theta + me\left(\frac{d^2x}{dt^2}\right)\sin\theta \tag{4}$$

where $I = I_o + me^2$.

It is convenient at this point to reduce Eqs. 1 and 2 to dimensionless form by dividing by ek_1, and Eq. 4 to dimensionless form by dividing by e^2k_1.

Then Eq. 1 becomes:

*Figures begin on page 23.

$$\frac{m}{k_1}\frac{d^2}{dt^2}\left(\frac{x}{e}\right) + \frac{c_1}{k_1}\frac{d}{dt}\left(\frac{x}{e}\right) + \frac{x}{e} = \frac{m}{k_1}\left[\left(\frac{d\theta}{dt}\right)^2 \cos\theta + \left(\frac{d^2\theta}{dt^2}\right)\sin\theta\right] \tag{5}$$

Eq. 2 becomes:

$$\frac{m}{k_1}\frac{d^2}{dt^2}\left(\frac{y}{e}\right) + \frac{c_2}{k_1}\frac{d}{dt}\left(\frac{y}{e}\right) + \frac{k_2}{k_1}\left(\frac{y}{e}\right) = \frac{m}{k_1}\left[\left(\frac{d\theta}{dt}\right)^2 \sin\theta - \left(\frac{d^2\theta}{dt^2}\right)\cos\theta\right] \tag{6}$$

and Eq. 4 becomes:

$$\left(\frac{I}{e^2 k_1}\right)\frac{d^2\theta}{dt^2} = \frac{M}{e^2 k_1} + \frac{m}{e k_1}\left[\left(\frac{d^2x}{dt^2}\right)\sin\theta - \left(\frac{d^2y}{dt^2}\right)\cos\theta\right] \tag{7}$$

Let: ζ = dimensionless deflection in the x direction = $\frac{x}{e}$

η = dimensionless deflection in the y direction = $\frac{y}{e}$

ξ = dimensionless time = $\sqrt{\frac{k_1}{m}}\, t$

$\frac{d^2\zeta}{dt^2} = \frac{d^2\zeta}{d\xi^2}\left(\frac{d\xi}{dt}\right)^2 = \ddot{\zeta}\,\frac{k_1}{m}$, where the dot denotes differentiation with respect to dimensionless time ξ.

Then Eq. 5 becomes:

$$\ddot{\zeta} + \frac{c_1}{\sqrt{m k_1}}\dot{\zeta} + \zeta = (\dot{\theta})^2 \cos\theta + \ddot{\theta}\sin\theta \tag{8}$$

Eq. 6 becomes:

$$\ddot{\eta} + \frac{c_2}{\sqrt{m k_1}}\dot{\eta} + \frac{k_2}{k_1}\eta = (\dot{\theta})^2 \sin\theta - \ddot{\theta}\cos\theta \tag{9}$$

and Eq. 7 becomes, after multiplication by $\frac{me^2}{I}$:

$$\ddot{\theta} = \frac{Mm}{I k_1} + \frac{me^2}{I}\left[\ddot{\zeta}\sin\theta - \ddot{\eta}\cos\theta\right] \tag{10}$$

Examining the damping terms in Eqs. 8 and 9, let:

c_{1c} = critical damping in the x direction = $2\sqrt{m k_1}$

c_{2c} = critical damping in the y direction = $2\sqrt{m k_2}$

Substituting, Eq. 8 becomes:

$$\ddot{\xi} + \left(2\frac{c_1}{c_{1c}}\right)\dot{\xi} + \xi = (\dot{\theta})^2 \cos\theta + \ddot{\theta}\sin\theta \tag{11}$$

and since:

$$c_{2c} = 2\sqrt{m k_2} = 2\sqrt{m k_1}\sqrt{\frac{k_2}{k_1}},$$

$$\sqrt{m k_1} = \frac{c_{2c}}{2}\sqrt{\frac{k_1}{k_2}}$$

Eq. 9 becomes:

$$\ddot{\eta} + \left(2\frac{c_2}{c_{2c}}\right)\sqrt{\frac{k_2}{k_1}}\,\dot{\eta} + \frac{k_2}{k_1}\eta = (\dot{\theta})^2 \sin\theta - \ddot{\theta}\cos\theta \tag{12}$$

Taking the damping ratios in the x and y directions to be equal (i.e., $c_1/c_{1c} = c_2/c_{2c}$), it is apparent from an examination of Eqs. 10, 11 and 12 that four independent dimensionless parameters define the problem, namely:

$$P(1) = \frac{m e^2}{I}$$

$$P(2) = \frac{M m}{I k_1}$$

$$P(3) = \frac{k_2}{k_1}$$

$$P(4) = \left(2\frac{c_1}{c_{1c}}\right) = \left(2\frac{c_2}{c_{2c}}\right)$$

where k_2 is equal to or less than k_1. Performing the above substitutions, Eq. 10 becomes:

$$\ddot{\theta} = P(2) + P(1)\left[\ddot{\xi}\sin\theta - \ddot{\eta}\cos\theta\right] \tag{13}$$

Eq. 11 becomes:

$$\ddot{\xi} + P(4)\,\dot{\xi} + \xi = (\dot{\theta})^2 \cos\theta + \ddot{\theta}\,\sin\theta \tag{14}$$

and Eq. 12 becomes:

$$\ddot{\eta} + P(4)\sqrt{P(3)}\,\dot{\eta} + P(3)\,\eta = (\dot{\theta})^2 \sin\theta - \ddot{\theta}\cos\theta \tag{15}$$

Equations 13, 14 and 15 are the dimensionless equations of motion which will now be arranged in suitable form for solution by numerical methods.

3. Numerical Solution.

The Runge-Kutta fourth order method of numerical integration was selected to solve the equations of motion. This choice was made on the basis of several considerations, namely: it is applicable to non-linear differential equations; it is "self-staring", i.e., only the functional values at a single previous point are required to obtain the functional values ahead; it is relatively simple to program; finally, since it is essentially the Taylor series solution through terms of order h^4, it offers an acceptable degree of accuracy for the solution of the problem. Before proceding, however, two additional parameters will be defined in order to provide the equations of motion with greater flexibility and to reduce the computation time.

The torque parameter P(2) is a constant for most of the presentation. However, to provide for the case where it is a function of speed, let P(2) be replaced by P(2)P(5), where P(5) is either unity for the constant torque cases, or a function of speed for the variable torque case.

Also, let:

$$P(6) = \sqrt{P(3)} = \sqrt{\frac{k_2}{k_1}}$$

In order to employ the Runge-Kutta method of numerical integration, it is necessary to reduce the three equations of motion to a system of six first order differential equations of the form:

$$\frac{dy_i}{dx} = f(x, y_1(x), y_2(x), \cdots, y_n(x))$$

Let:

$$\dot{\theta} = \alpha \quad (16)$$

$$\dot{\eta} = \beta \quad (17)$$

$$\dot{\xi} = \gamma \quad (18)$$

Substituting these variables and the parameters P(2)P(5) and P(6) in the equations of motion, Eq. 13 becomes:

$$\dot{\alpha} = P(2)P(5) + P(1)\left[\dot{\gamma} \sin\theta - \dot{\beta}\cos\theta\right] \tag{19}$$

Eql 14 becomes:

$$\dot{\gamma} + P(4)\gamma + \zeta = \alpha^2 \cos\theta + \dot{\alpha}\sin\theta \tag{20}$$

and Eq. 15 becomes:

$$\dot{\beta} + P(4)P(6)\beta + P(3)\eta = \alpha^2 \sin\theta - \dot{\alpha}\cos\theta \tag{21}$$

Equations 16 through 21 constitute the system of first order differential equations which must be reduced to the form:

$$\frac{dy_i}{dx} = f(x, y_1(x), y_2(x), \cdots, y_n(x))$$

Appendix I contains this rearrangement. The resulting equations are:

$$\dot{\theta} = \alpha \tag{16}$$

$$\dot{\eta} = \beta \tag{17}$$

$$\dot{\zeta} = \gamma \tag{18}$$

$$\dot{\alpha} = \Big[P(2)P(5) + P(1)P(4)P(6)\beta\cos\theta + P(1)P(3)\eta\cos\theta - P(1)P(4)\gamma\sin\theta - P(1)\zeta\sin\theta\Big] \Big/ \Big[1 - P(1)\Big] \tag{22}$$

$$\dot{\beta} = \Big[-P(1)\alpha^2 \sin\theta\cos^2\theta + P(1)P(4)\gamma\sin\theta\cos\theta + P(1)\zeta\sin\theta\cos\theta + \alpha^2\sin\theta - P(4)P(6)\beta - P(3)\eta - P(2)P(5)\cos\theta - P(1)\alpha^2\sin^3\theta + P(1)P(4)P(6)\beta\sin^2\theta + P(1)P(3)\eta\sin^2\theta\Big] \Big/ \Big[1 - P(1)\Big] \tag{23}$$

$$\dot{\gamma} = \Big[\alpha^2 \cos\theta - P(4)\,\gamma - \varsigma - P(1)\,\alpha^2 \sin^2\theta \cos\theta + \qquad (24)$$

$$P(1)\,P(4)\,P(6)\,\beta \sin\theta \cos\theta + P(1)\,P(3)\,\eta \sin\theta \cos\theta -$$

$$P(1)\,\alpha^2 \cos^3\theta + P(1)\,P(4)\,\gamma \cos^2\theta + P(1)\,\varsigma \cos^2\theta +$$

$$P(2)\,P(5) \sin\theta\Big] \Big/ \Big[1 - P(1)\Big]$$

Having selected the integration method and reduced the equations of motion to the proper form for solution by this method, the program itself will now be considered.

4. Computer Program Design and Use.

Several factors governed the form of the computer program. In the material that follows, the more important of these are discussed in detail, while those of lesser importance are mentioned briefly. Details can be found in the block diagram or in the program itself, Appendices III and IV, respectively.

Failure to accelerate through the critical speed region can result from either of two closely related conditions, insufficient driving torque or excessive unbalance of the rotor. An insufficient driving torque results in a condition wherein the angular velocity of the rotor remains in the vicinity of that corresponding to its natural frequency of vibration for a long period of time. In this case the inertia forces, though initially small, eventually produce large lateral deflections which in turn develop counter torques opposing the driving torque. An excessive unbalance produces large inertia forces as a result of the greater eccentricity of the mass center. The resulting lateral deflections also produce counter torques opposing the applied torque. It follows, then, that the success or failure of an attempted acceleration through the critical speed region is dependent upon the applied torque and the eccentricity of the mass center, other physical quantities remaining constant. Hence a plot of the eccentricity of the mass center versus the applied torque for various acceleration attempts will indicate points of either successful or unsuccessful accelerations through the critical speed regions. A boundary line may then be drawn to separate those points representing successful accelerations from those points representing unsuccessful accelerations.

For a given system, the parameter P(1) is directly proportional to the square of the eccentricity of the mass center, while parameter P(2) is directly proportional to the applied torque. These parameters, the eccentricity parameter P(1) and the torque parameter P(2), are the ordinate and

abscissa, respectively, of the curve which is the boundary between the areas of successful and unsuccessful accelerations. The boundary itself is defined by pairs of points. Each pair of points represents two acceleration attempts, one successful, the other unsuccessful, differing by five per cent of the torque parameter P(2), all other parameters being constant. It is now appropriate to define a successful and an unsuccessful acceleration.

Referring to Eqs. 20 and 21, one of the natural circular frequencies of vibration of the dimensionless system is unity and the other is the square root of the stiffness ratio, $\sqrt{\frac{k_2}{k_1}}$. The maximum value of the latter is unity. On this basis, a successful acceleration was defined as an acceleration in which a (dimensionless) angular velocity of two (twice that corresponding to the higher natural circular frequency of vibration) was reached. That is, if a dimensionless angular velocity of two was reached, the acceleration through the critical speed region was considered successful and the program terminated.

The possibility existed, however, that during the fluctuations in the angular velocity of an otherwise unsuccessful acceleration a speed of two might have been momentarily attained, upon which the program would have terminated and the acceleration would have been erroneously termed successful. To investigate this possibility, several random successful and unsuccessful accelerations were initiated, and graphs of dimensionless angular velocity versus dimensionless time were made. Two such curves can be seen in Fig. 3. For the successful acceleration the torque parameter is five per cent larger than for the unsuccessful acceleration, all other parameters being unchanged. It is apparent that the maximum angular velocity for the unsuccessful acceleration is well below that value (2.0) which would terminate the program and falsely indicate a successful acceleration.

An unsuccessful acceleration was defined essentially in terms of a successful acceleration. Starting from a known successful acceleration the torque parameter P(2) was decreased in five per cent increments, while the time to reach a dimensionless angular velocity of two was recorded. In regions far removed from the boundary between the successful and unsuccessful acceleration regions, the time required for succeeding successful accelerations increased by five per cent. In the vicinity of the boundary, however, the additional time required for a successful acceleration exceeded the expected five per cent, but in no case did it exceed thirty per cent. On this basis, the time allowed for each successive acceleration was twice that required by the previous successful acceleration. If a dimensionless angular velocity of two was not achieved within this period of time, the attempt was considered unsuccessful and the program terminated. Figure 3 also illustrates this point. The upper curve represents a successful acceleration in which a dimensionless angular velocity of two was reached in time t. The lower curve represents an unsuccessful acceleration in which a dimensionless angular velocity of two was not achieved in time 2t. The torque parameters for these attempts differ by five per cent, all other parameters being constant. Hence these two attempts correspond to one of the several pairs of points which define the boundary between regions of successful and unsuccessful accelerations.

For the undamped case, an energy balance affords a check on the validity of the results. At any given time the total energy supplied to the system must equal the sum of the kinetic energies of rotation and translation, and the potential energy stored in the springs, or:

$$M\theta = \frac{1}{2} I_o \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} m \left(\frac{d x_o}{dt}\right)^2 + \frac{1}{2} m \left(\frac{d y_o}{dt}\right)^2 + \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 \qquad (25)$$

An equivalent statement in dimensionless form is:

$$F(2)\,\theta = \frac{P(1)}{2}\left[\frac{(\dot{\theta})^2}{P(1)} + (\dot{\xi})^2 + (\dot{\eta})^2 + \xi^2 + P(3)\,\eta^2 - 2\dot{\theta}(\xi \sin\theta - \eta \cos\theta)\right] \tag{26}$$

For the undamped runs, this check was performed at the end of each acceleration; i.e., employing the terminal values of the quantities in Eq. 26.

The choice of a suitable time increment was made on the basis of a comparison of the results from several solutions of the same problem employing various time increments. Details are given in Appendix II. A (dimensionless) time increment of 0.10 was selected.

Since the objectives of this investigation included a determination of the maximum amplitudes of vibration upon acceleration through the critical speed region, this quantity, the resultant of the deflections in the x and y directions, was computed each time increment and compared with a stored maximum which had been determined earlier, the larger being retained.

Provision was also made to terminate the solution at any point and to resume the solution at this same point at a later date. This innovation provided a considerable savings in computer time since any time period, however small, could be completely utilized, and no solutions were lost because of unexpected requests to release the computer. The termination and resumption was executed by writing the values of the (dimensionless) deflections and velocities in the x and y directions, the angular velocity and displacement and elapsed time on magnetic tape in binary form, and reading these same values back into the program by the setting of certain selective jump switches.

By way of a check of the computer, the output tape unit and the tape itself, the same short run was performed at the beginning of each period of computer use.

The program output included the parameters employed, the results of the energy balance, the maximum angular velocity and the time of its occurrence, and the final values of the deflections and velocities in the x and y directions, angular velocity, revolutions, and elapsed time. Also recorded were the maximum amplitudes of vibration in the x direction, y direction and the maximum resultant amplitude. For each of the three maximum amplitudes of vibration, the time of its occurrence, angular velocity and the x direction, y direction and resultant amplitudes of vibration were also recorded.

In general, the investigation proceeded as follows. Selecting an appropriate set of values for parameters P(1), P(3) and P(4), the torque parameter P(2) was varied to define the boundary between the regions of successful and unsuccessful accelerations. Having defined the boundary, P(2) was then increased through an appropriate range of values to obtain the corresponding maximum amplitudes of vibration. This procedure was then repeated for different values of P(1), P(3) and P(4).

The following section presents the results of the investigation.

5. Discussion of Results.

The results of the investigation are presented as follows. First, the boundary between the regions of successful and unsuccessful acceleration is discussed for stiffness ratios of unity, 0.50, 0.75 and 0.25. Then the curves of the maximum dimensionless amplitudes of vibration are presented. Finally, the effects of added damping are considered.

A rotor having equal spring constants in the x and y directions (P(3) = 1.0) and no damping (P(4) = 0) was investigated first. Figure 4* indicates the location of the boundary between the regions of successful and unsuccessful accelerations as a function of the eccentricity parameter P(1) and the torque parameter P(2). This curve is essentially a reference for the boundary curves that follow since unequal stiffness and added damping are more conveniently discussed in terms of their effects on this particular result.

Figure 4 shows that at successively smaller eccentricities, smaller applied torques are required for successful accelerations, since the smaller resultant inertia forces produce smaller counter torques.

Decreasing the stiffness ratio to 0.50, the undamped boundary curve was found to be as shown in Fig. 5. For any non-unity stiffness ratio, the system will have two critical speed regions; in this particular case one is in the vicinity of a (dimensionless) angular velocity of $\sqrt{0.50}$ and the other is in the vicinity of unity. It follows, then, that an unsuccessful acceleration can result from an inability to accelerate through either the lower critical speed region or the higher critical speed region. Figure 5 illustrates both of these conditions. The boundary between the regions of successful and unsuccessful accelerations is seen to vary between two parallel limits. In the vicinity of the higher limit, the unsuccessful acceleration points defining the boundary represent failure to accelerate through the higher critical speed region. Similarly,

*Figure 4 can be found on page 26.

those unsuccessful accelerations defining the boundary in the vicinity of the lower limit represent failure to accelerate through the lower critical speed region.

The limits mentioned above have a definite relationship to each other. For a given eccentricity, the ratio of the lower limit to the upper limit is very nearly equal to the stiffness ratio, k_2/k_1. This can be explained by examining the dimensionless torque parameter $P(2) = \frac{Mm}{Ik_1}$. It is apparent from this dimensionless grouping that a system which has a spring constant of, say, Ak_1 will require a torque of AM for a given eccentricity. In the vicinity of the lower critical speed region of the system under consideration, the system behaves essentially as a single degree of freedom system having a stiffness $k_2 = 0.50k_1$. Hence the torque required for acceleration is 0.50M. Since the boundary curves are plotted versus P(2), which is a function of k_1, the substitution of 0.50M for M results in a decrease of the torque parameter from P(2) to 0.50P(2).

The fact that a portion of the boundary curve is in the vicinity of the lower limit shows that, in this region, a successful transit of the lower critical speed region will result in a successful transit of the higher critical speed region also. This condition may be due to a favorable phase relationship between those vibrations induced at the lower and higher critical speeds. That is, those vibrations remaining in the system as a result of the transit of the lower critical speed region exert a forward torque opposing that developed by the vibrations induced in the higher critical speed region to the extent that a smaller torque parameter suffices for a successful acceleration.

For a stiffness ratio of 0.75, the undamped boundary curve was found to be as shown in Fig. 6. The general shape of the curve is comparable to that of the stiffness ratio previously discussed, 0.50; however, some differences do exist.

The boundary is seen to vary between two limits much the same as the boundary of the 0.50 stiffness ratio case; however, for a given eccentricity, the ratio of the torque at the lower limit to that at the upper limit is influenced considerably by the close proximity of the two natural frequencies of vibration. In the higher torque regions, this influence is quite apparent, and as a result, the limit ratio differs somewhat from the stiffness ratio. In the lower torque regions, agreement between the stiffness ratio and the limit ratio is quite good since there are many revolutions of the rotor separating the critical speed regions and their interaction is reduced considerably.

For a stiffness ratio of 0.25, the critical speeds of 0.50 and unity are sufficiently displaced from each other that little or no interaction of vibrations is experienced. The counter torques developed in the higher critical speed region are larger than those induced in the lower critical speed region due to the larger angular velocity. It follows that an unsuccessful acceleration represents a failure to accelerate through the higher critical speed region. The unsuccessful acceleration points defining the boundary curve shown in Fig. 7 represent just such a failure.

By way of a brief summary, the undamped boundary curves described above, i.e., for stiffness ratios of 1.00, 0.50, 0.75 and 0.25, are plotted on a single graph, Fig. 8.

Except as noted below, the starting phase $\theta = 0$ was used. When the lateral stiffnesses are equal ($P(3) = 1$), the resulting circular symmetry assures that neither the location of the boundary nor the maximum vibration amplitude will be affected by starting phase. Even with unequal stiffness ratios, it is reasonable to expect that the effect of starting phase is negligible if the rotor completes a large number of revolutions before reaching the lower critical speed. This condition is met for small values of the torque parameter.

An anomalous behavior in the higher torque region did result in a limited study of the effect of starting phase. With a stiffness ratio of 0.50, it was found at $P(1) = 2.30 \times 10^{-3}$ that the minimum value of P(2) for successful acceleration was 46.0 per cent less than at immediately adjacent higher and lower values of P(1). Further runs were made at $P(1) = 2.30 \times 10^{-3}$ with starting values of $\theta = 45$, 90 and 135 degrees and it was found that the minimum P(2) for successful acceleration approached the results obtained for neighboring values of P(1). Because the principal engineering interest in these results is in selecting a combination of parameters that will assure a successful acceleration, regardless of starting phase, the "stray" point at $P(1) = 2.30 \times 10^{-3}$ obtained with $\theta = 0$ was ignored in plotting the boundary curve of Fig. 5.

Figures 9 through 12 are the undamped maximum amplitude curves for the stiffness ratios investigated. These curves indicate that, for a given eccentricity, successively larger applied torques result in decreasing amplitudes of vibration. These smaller amplitudes are to be expected since the larger torques result in a shorter period of time in the critical speed regions.

These curves further show that in regions of successful accelerations removed from the boundary between the regions of successful and unsuccessful accelerations, the maximum dimensionless amplitudes of vibration are independent of the eccentricity parameter P(1); or, for a given torque parameter P(2):

$$R/e = \text{Constant}$$

from which:

$$R = (e)\ (\text{Constant})$$

Therefore, the maximum amplitude of vibration is directly proportional to the eccentricity e of the mass center. This is to be expected since these vibrations are a consequence of the inertia forces which, in turn are directly proportional to the eccentricity of the mass center.

The unusual variations in the amplitude curves of Figs 10 and 11 are a result of the transition of the boundary curve from one of its limits to the other limit.

Figures 13 through 20 indicate the effects of the addition of one per cent of critical damping on the boundary and amplitude curves. With one exception, shown in Fig. 15, these curves reveal a decrease in the maximum amplitudes and a decrease in the torque required for a successful acceleration. The decreased amplitudes result in smaller counter torques, hence acceleration through the critical speed region is achieved with a smaller applied torque. The larger shift of the boundary in the lower torque regions is a result of the larger number of cycles during which the damping has acted.

Figure 15, however, shows that the effects of added damping are unfavorable for a considerable range of values of eccentricity and torque parameters. It is possible that the added damping, though reducing the individual amplitudes of vibration induced in the lower and higher critical speed regions, could have also resulted in a particularly unfavorable phase relationship between these vibrations. As a consequence the resulting amplitudes and counter torques could have been particularly large and a larger applied torque would be required for successful acceleration.

As was the case with Figs. 10 and 11, the unusual variations of the amplitude curves of Figs. 18 and 19 are a result of the transition of the boundary curve from one of its limits to the other.

6. Sample Problem.

The computer program is not limited to accelerations with constant applied torques. The following is a sample problem illustrating the extension of the solution to a system having a variable, rather than a constant, applied torque.

An unbalanced rotor, its supporting shaft and drive sheave have a combined mass m = 12.0 lbm, and a moment of inertia about the mass center I_o = 48.0 lbm-in^2. The mass center of the system is displaced a distance e = 0.02 in. from the center of rotation. The stiffnesses of the shaft and its supports are the same in the x and y directions and are $k_1 = k_2 = 100$ lbf/in. The drive motor for this system has a starting torque M = 40.0 lbf-in. which decreases linearly with angular velocity to a value of zero at 55.2 rpm, or twice the critical speed of the system. If damping is negligible, determine the maximum amplitude of vibration.

The axis of the moment of inertia must be translated from the mass center to the center of rotation. Since:

$$I = I_o + me^2 ,$$

$$I = (48.0 \text{ lbm-in}^2) + (12.0 \text{ lbm})(0.02 \text{ in.})^2 \cong 48.0 \text{ lbm-in}^2$$

Defining the parameters:

$$P(1) = \frac{me^2}{I}$$

$$P(1) = \frac{(12.0 \text{ lbm})(.02 \text{ in.})^2}{(48.0 \text{ lbm in}^2)} = 1.00 \times 10^{-4}$$

Similarly:

$$P(2) = \frac{Mm}{I k_1}$$

$$P(2) = \frac{(40.0 \text{ lbf-in.})(12.0 \text{ lbm})}{(48.0 \text{ lbm-in}^2)(100 \text{ lbf/in.})} = 0.100$$

This value of torque decreases linearly to zero at a (dimensionless) angular velocity of 2.00 (twice critical speed), therefore the modifying parameter P(5), which has been unity for the constant torque case, now becomes:

$$P(5) = \left(1 - \frac{\dot{\theta}}{2}\right)$$

and the dimensionless torque terms in the equations of motion become:

$$P(2)\ P(5) = 0.100\left(1 - \frac{\dot{\theta}}{2}\right)$$

Also:

$$P(3) = \frac{k_2}{k_1} = 1.00$$

and, since damping is negligible:

$$P(4) = 0.00$$

Inserting these parameters into the computer program, the resulting solution indicates:

$$\frac{R}{e} = 7.49$$

from which:

$$R = 0.160 \text{ in.}$$

where R is the maximum resultant amplitude of vibration.

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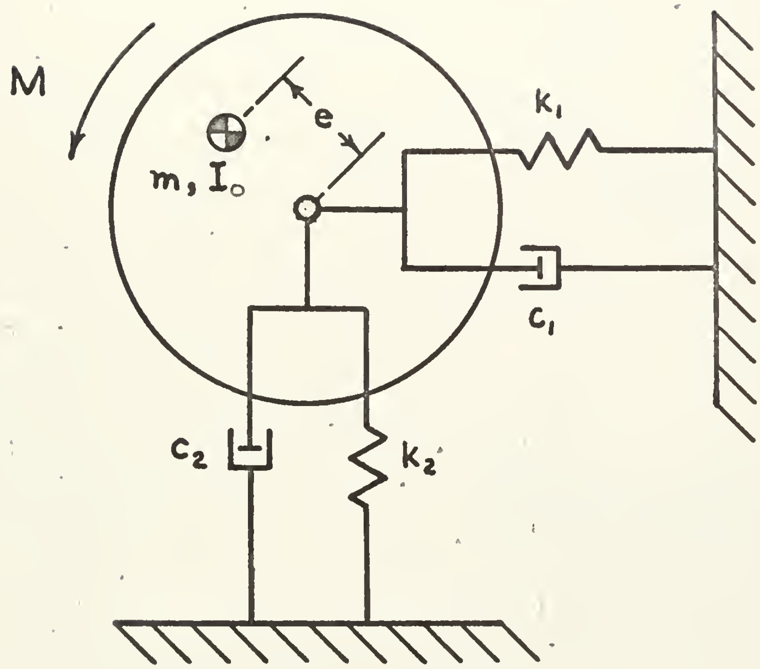


Fig. 1 Idealized system

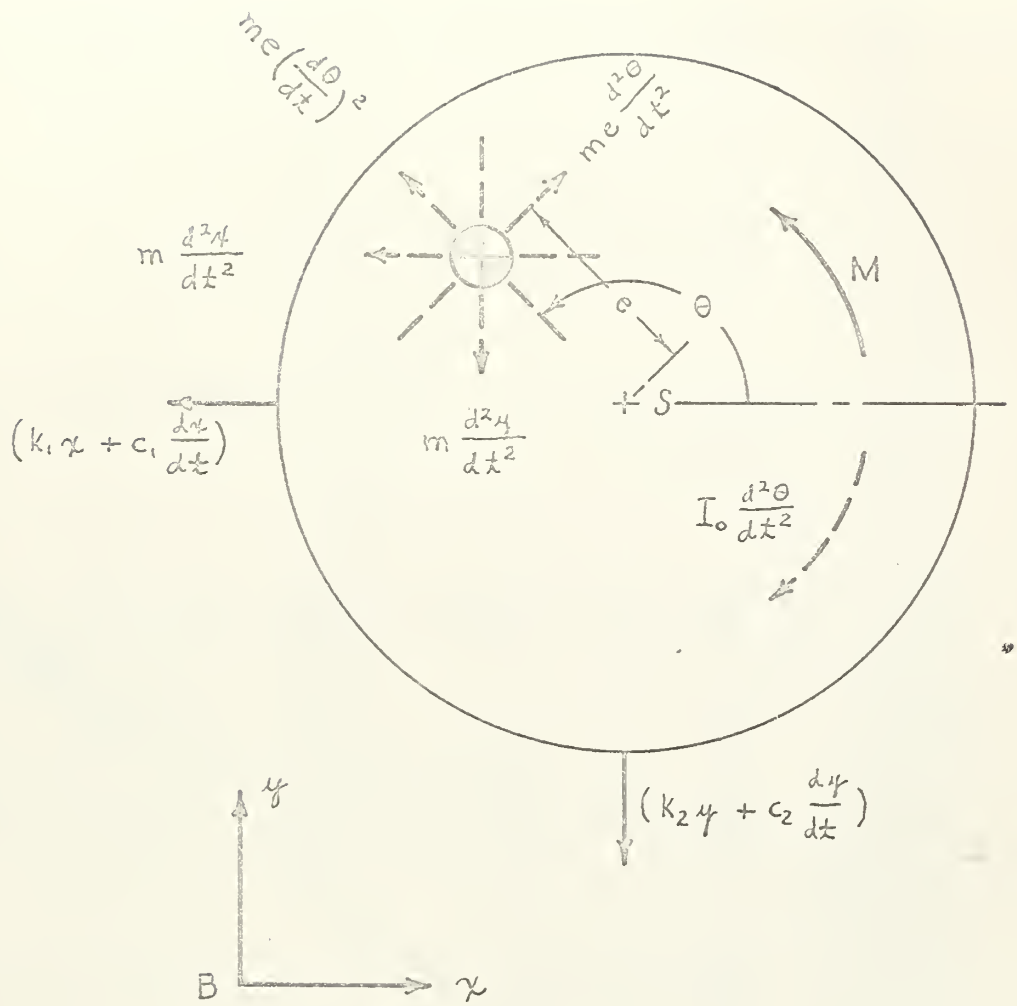


Fig. 2 Free body diagram

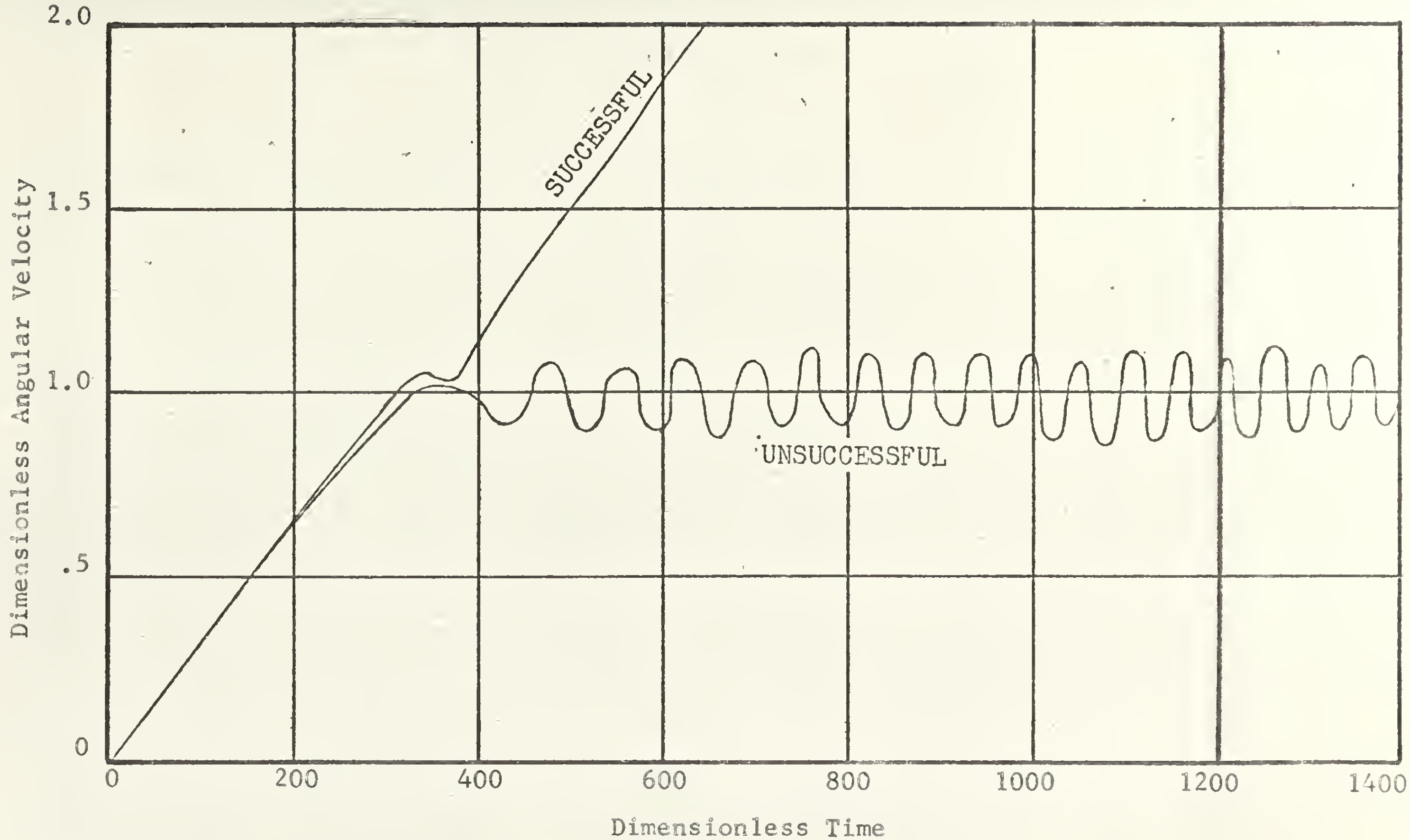


Fig. 3 Plot of dimensionless angular velocity versus dimensionless time for a successful and an unsuccessful acceleration. The successful acceleration has a torque parameter five per cent larger than that of the unsuccessful acceleration, all other parameters being unchanged.

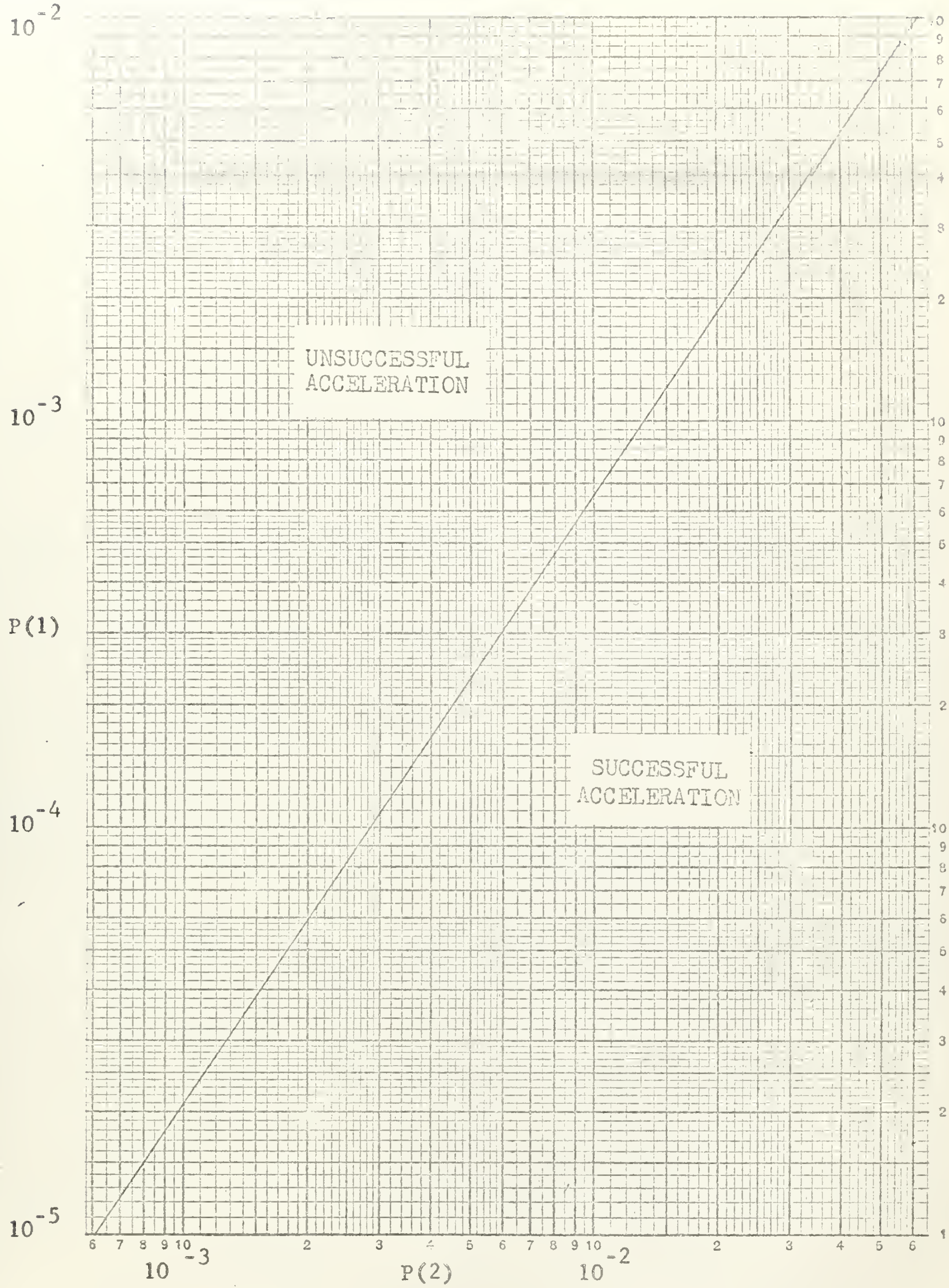


Fig. 4 Location of the boundary between the regions of successful and unsuccessful accelerations, P(3) = 1.0 (unity stiffness ratio), P(4) = 0 (undamped case).

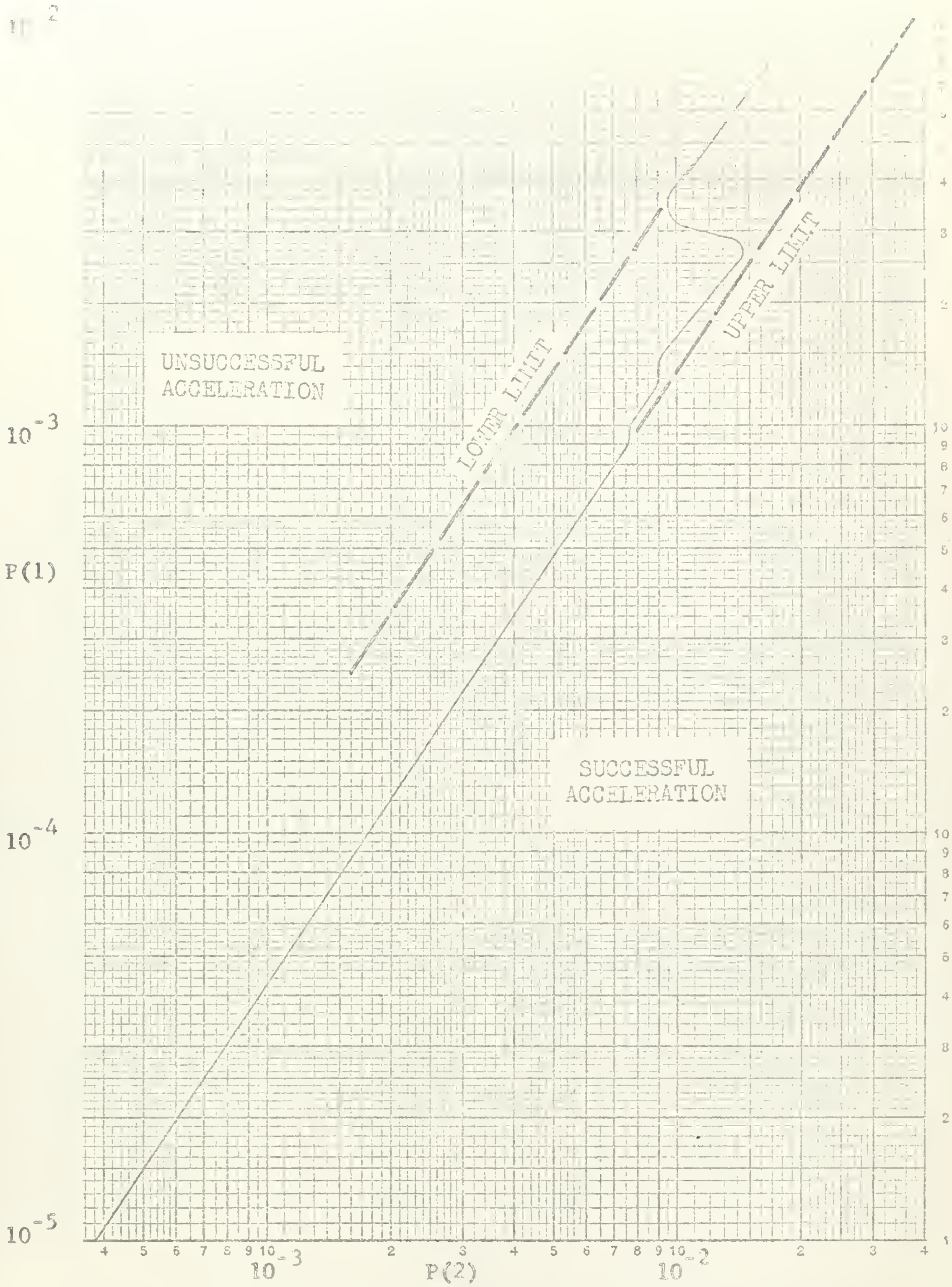


Fig. 5 Location of the boundary between the regions of successful and unsuccessful accelerations for a stiffness ratio P(3) = 0.50, P(4) = 0 (undamped case).

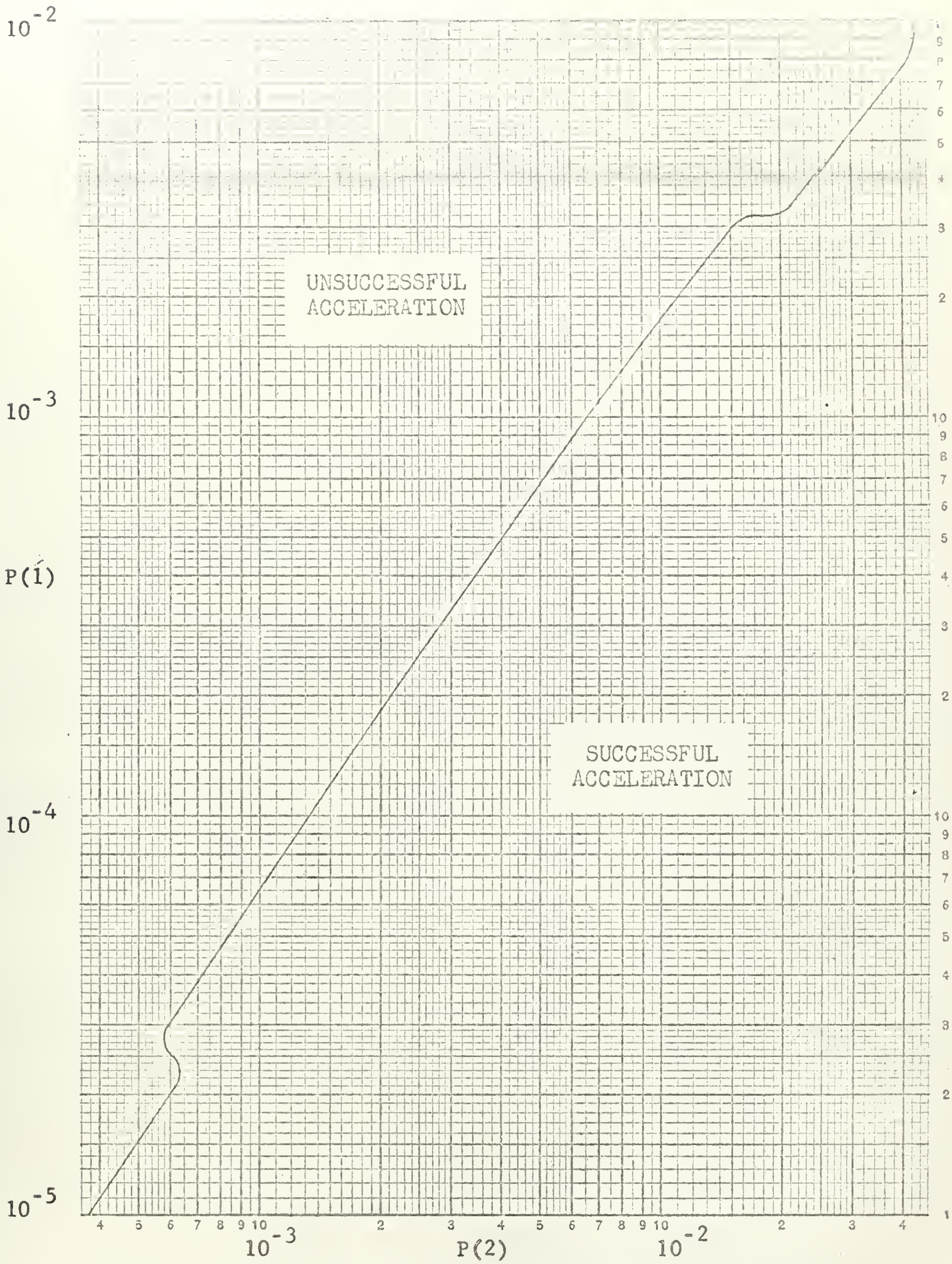


Fig. 6 Location of the boundary between the regions of successful and unsuccessful accelerations for a stiffness ratio P(3) = 0.75, P(4) = 0 (undamped case).

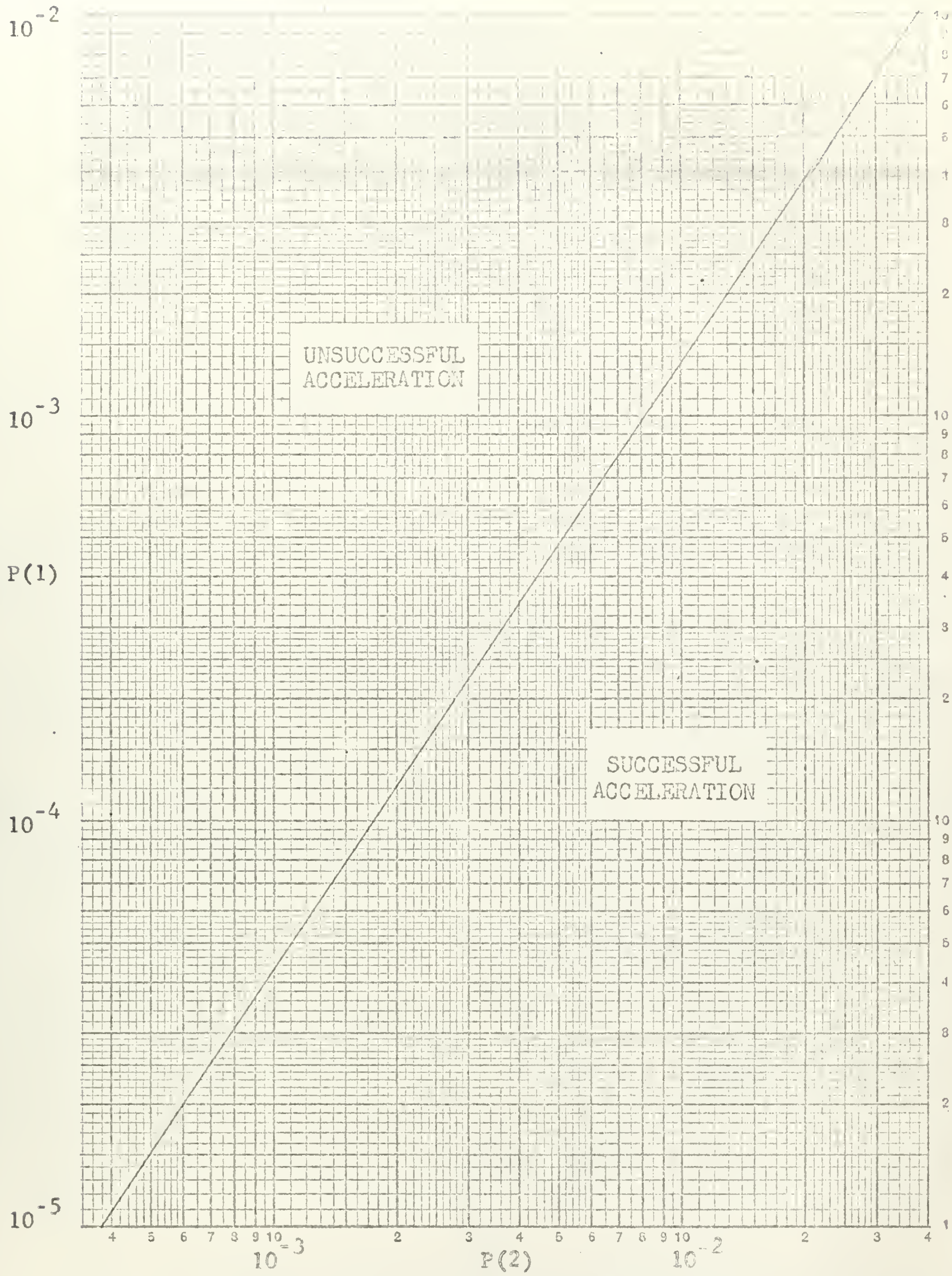


Fig. 7 Location of the boundary between the regions of successful and unsuccessful accelerations for a stiffness ratio P(3) = 0.25, P(4) = 0 (undamped case).

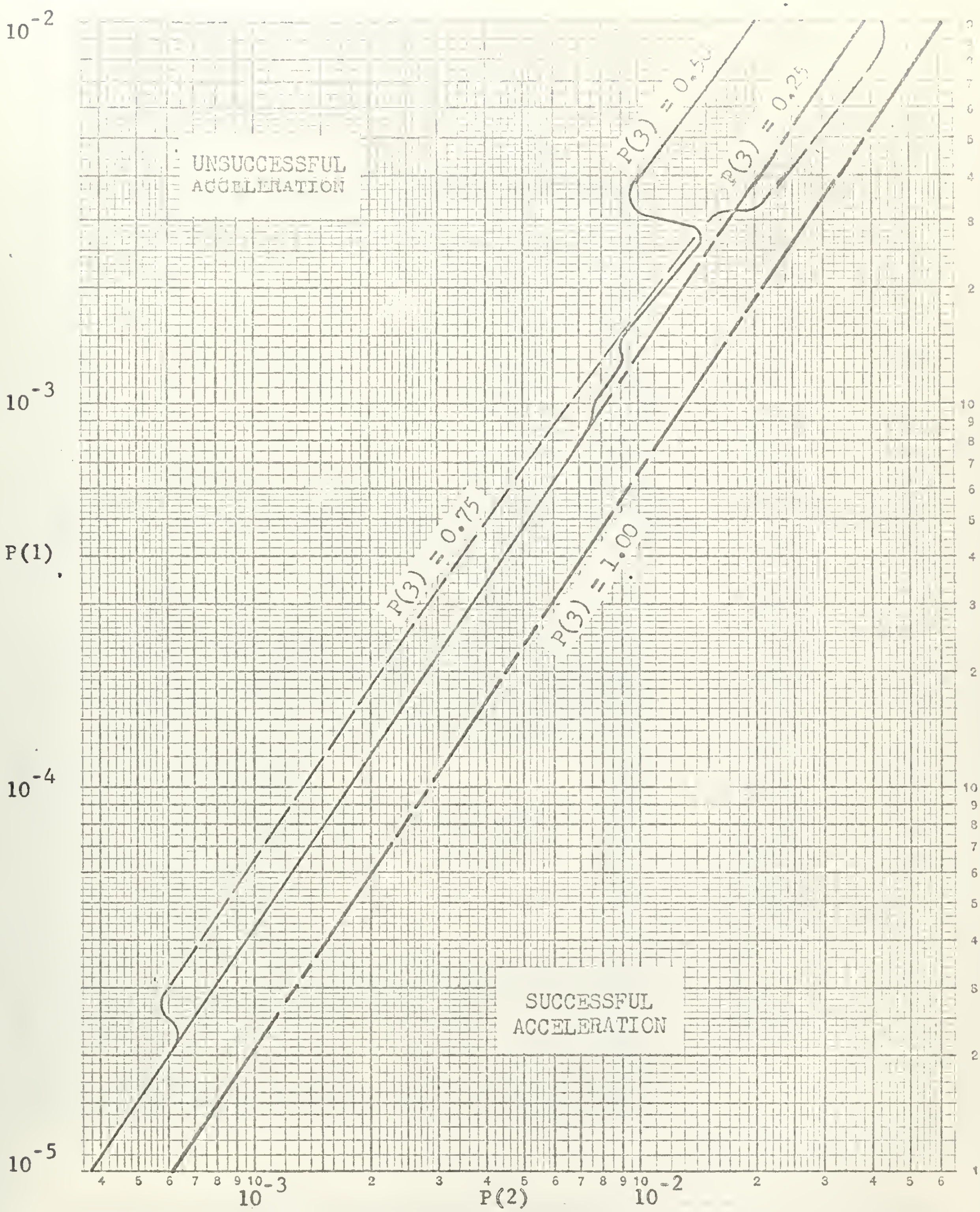


Fig. 8 Location of the boundary between the regions of successful and unsuccessful accelerations for stiffness ratios of 1.00, 0.75, 0.50 and 0.25, P(4) = 0 (undamped cases).

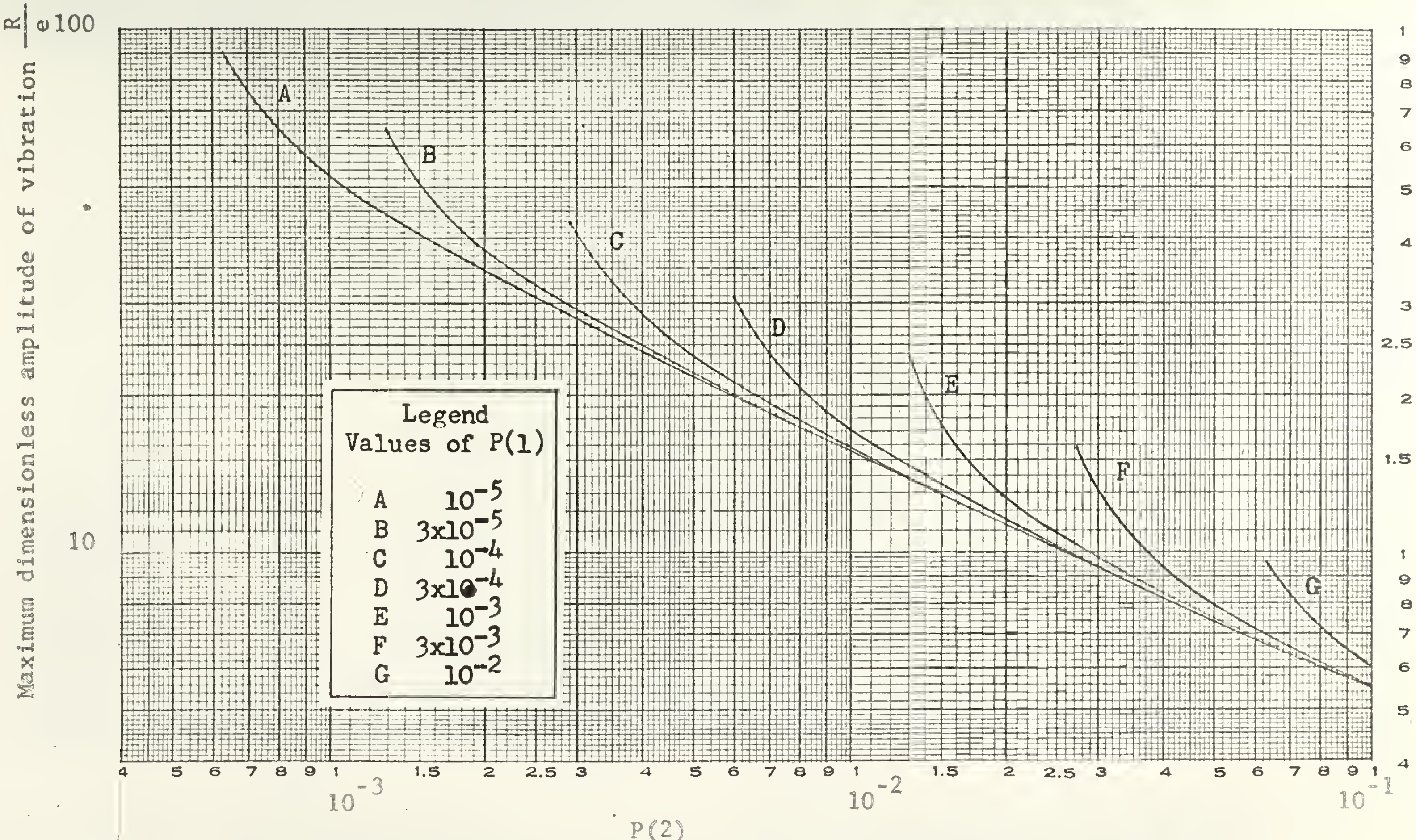


Fig. 9 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2); stiffness ratio P(3) = 1.0, P(4) = 0 (undamped case).

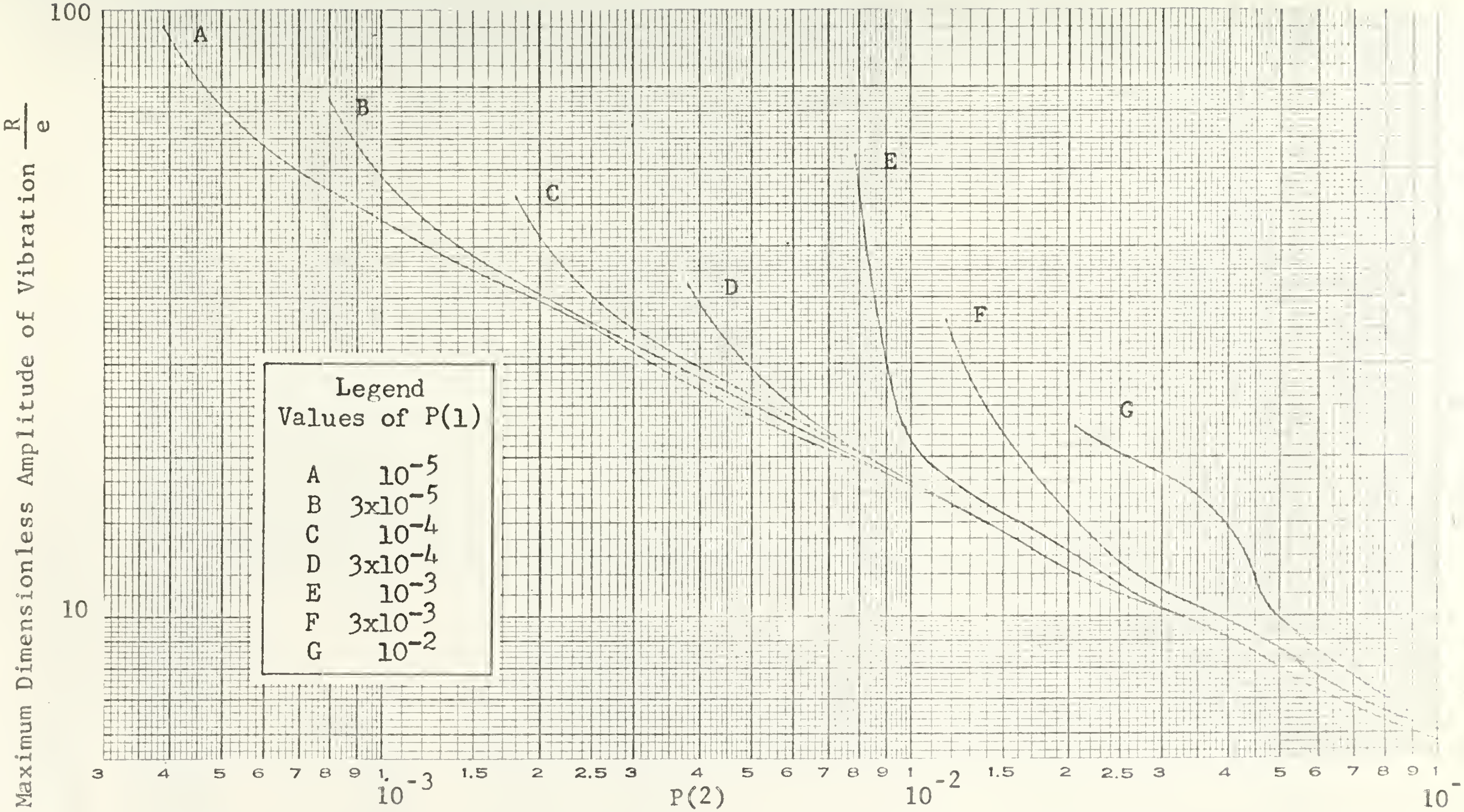


Fig. 10 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2); stiffness ratio P(3) = 0.50, P(4) = 0 (undamped case).

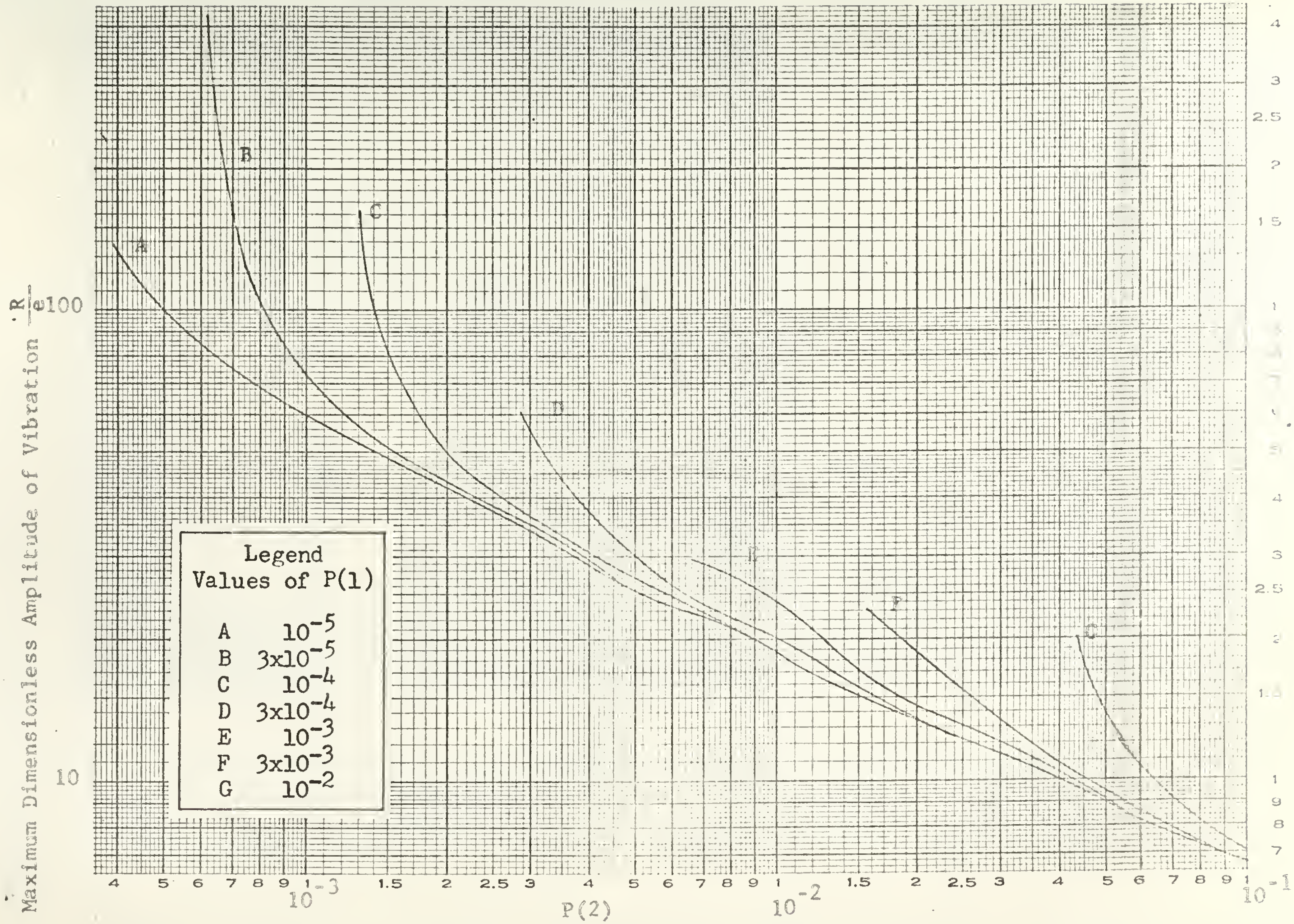


Fig. 11 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), stiffness ratio P(3) = 0.75, P(4) = 0 (undamped case).

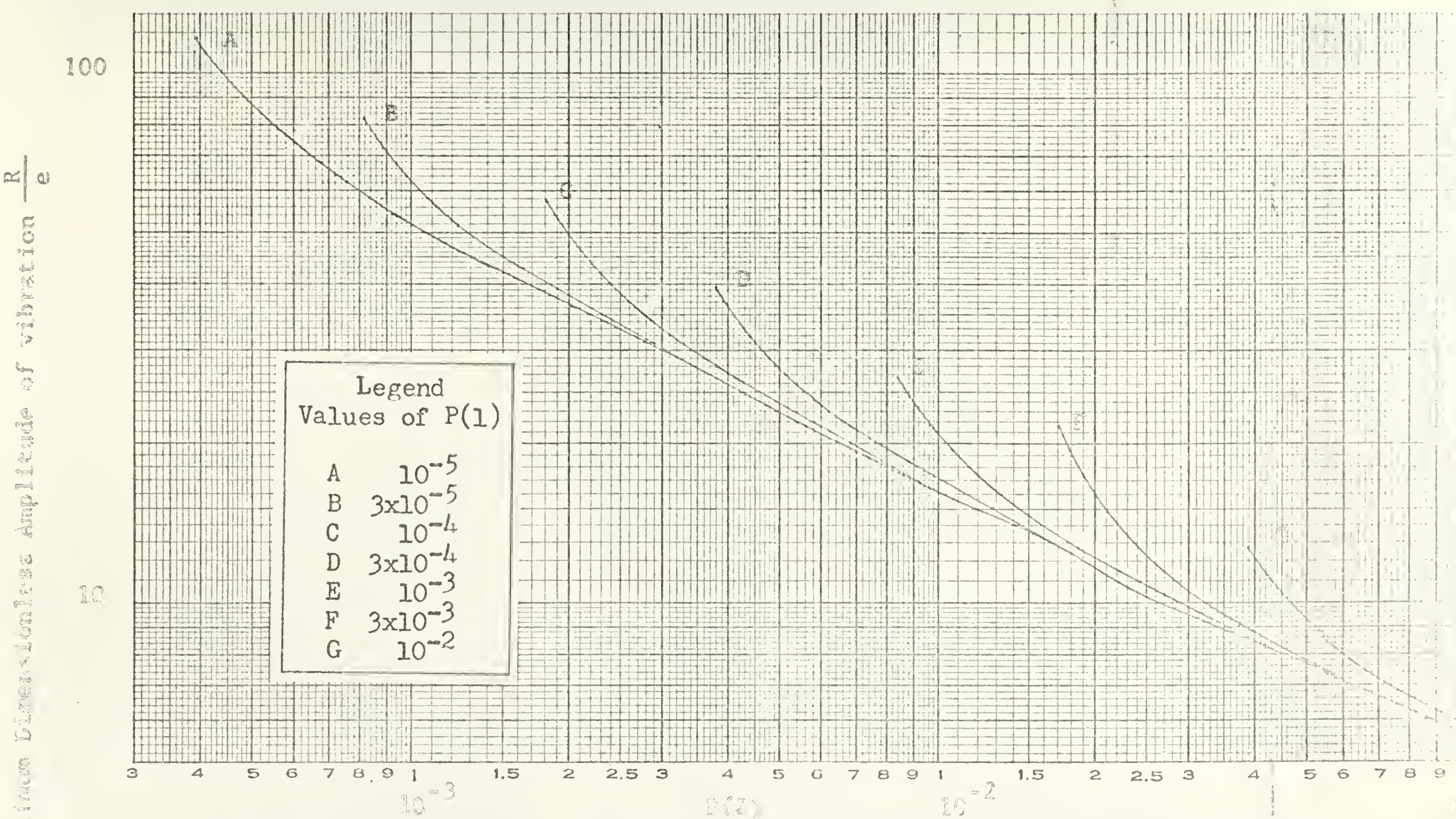


Fig. [illegible] Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), stiffness ratio = 0.25, P(4) = 0 (undamped case).

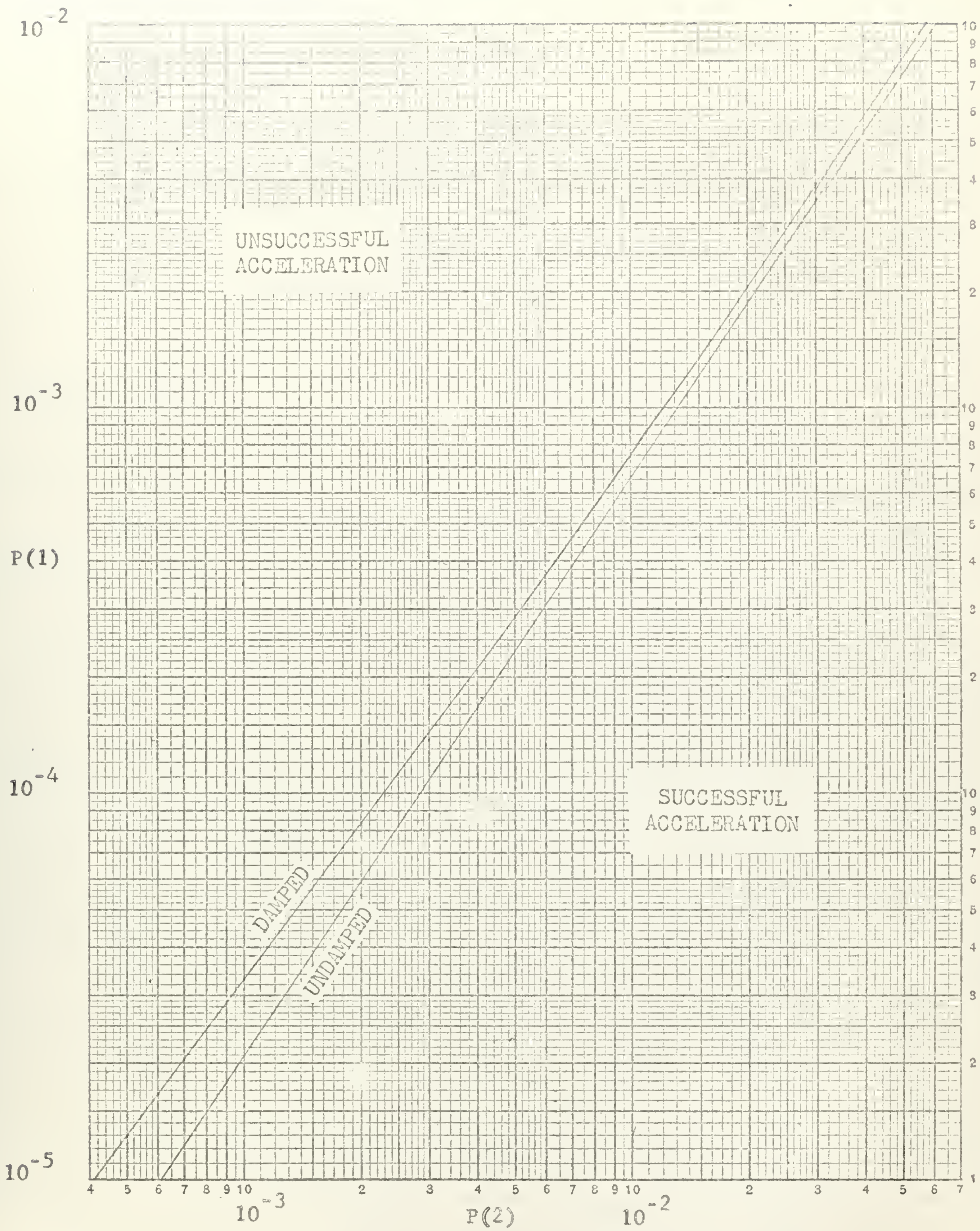


Fig. 13 Effect of 1.00 per cent of critical damping on the location of the boundary between regions of successful and unsuccessful accelerations, stiffness ratio P(3) = 1.00, P(4) = 0.02.

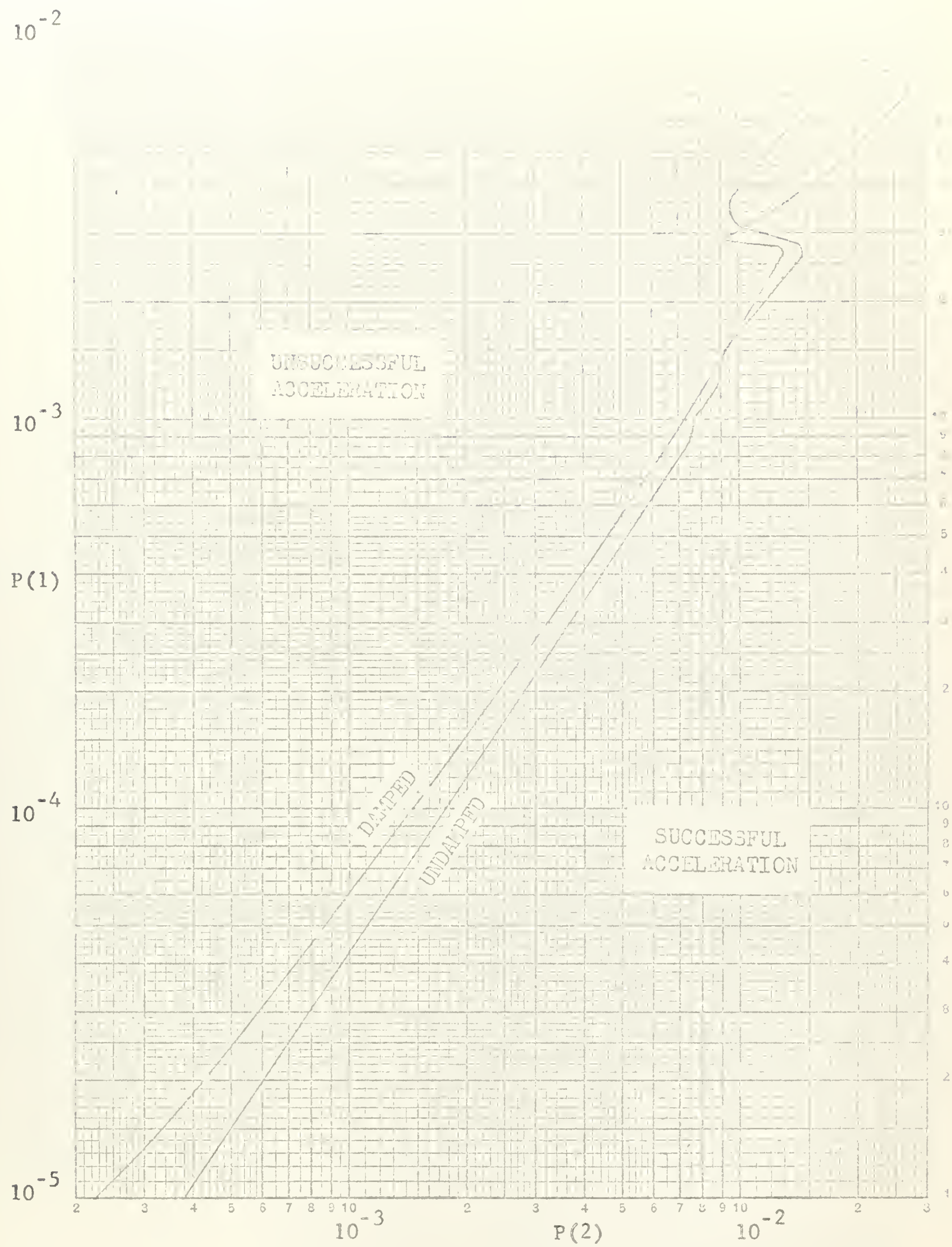


Fig. 14 Effect of 1.00 per cent of critical damping on the boundary between the regions of successful and unsuccessful accelerations, stiffness ratio P(3) = 0.50, P(4) = 0.02.

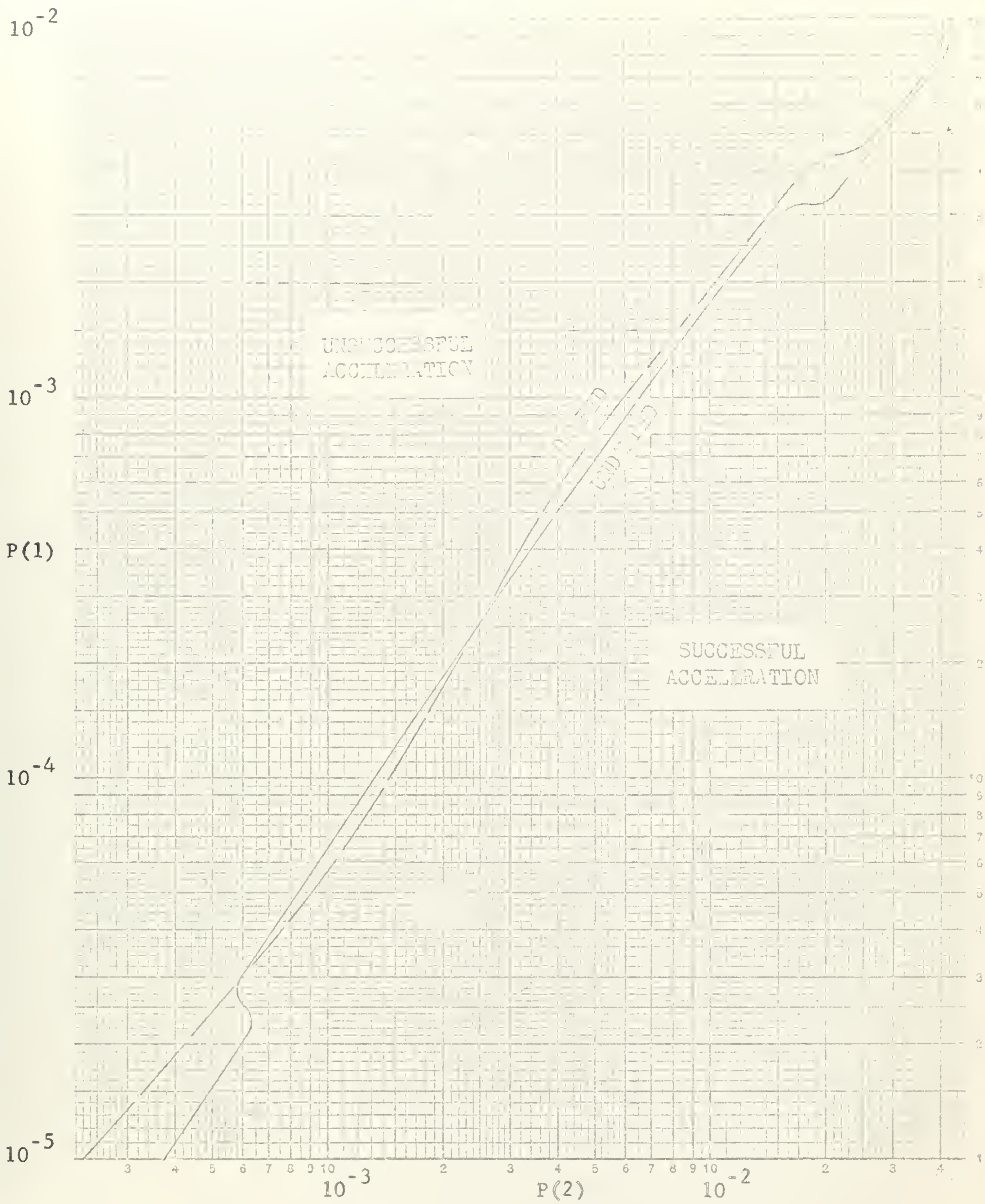


Fig. 15 Effect of 1.00 per cent of critical damping on the boundary between the regions of successful and unsuccessful accelerations, stiffness ratio P(3) = 0.75, P(4) = 0.02.

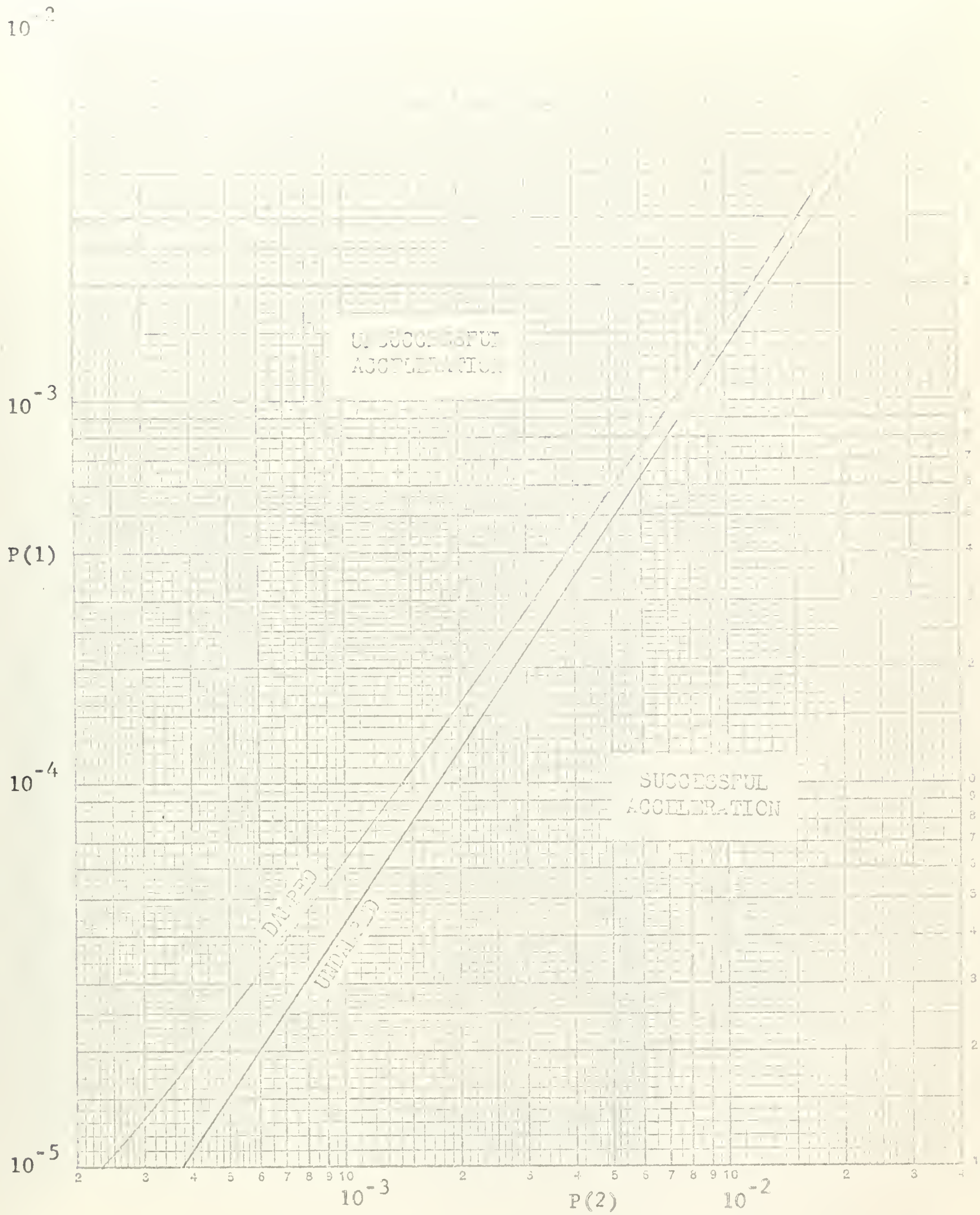


Fig. 16 Effect of 1.00 per cent of critical damping on the boundary between the regions of successful and unsuccessful accelerations, stiffness ratio P(3) = 0.25, P(4) = 0.02.

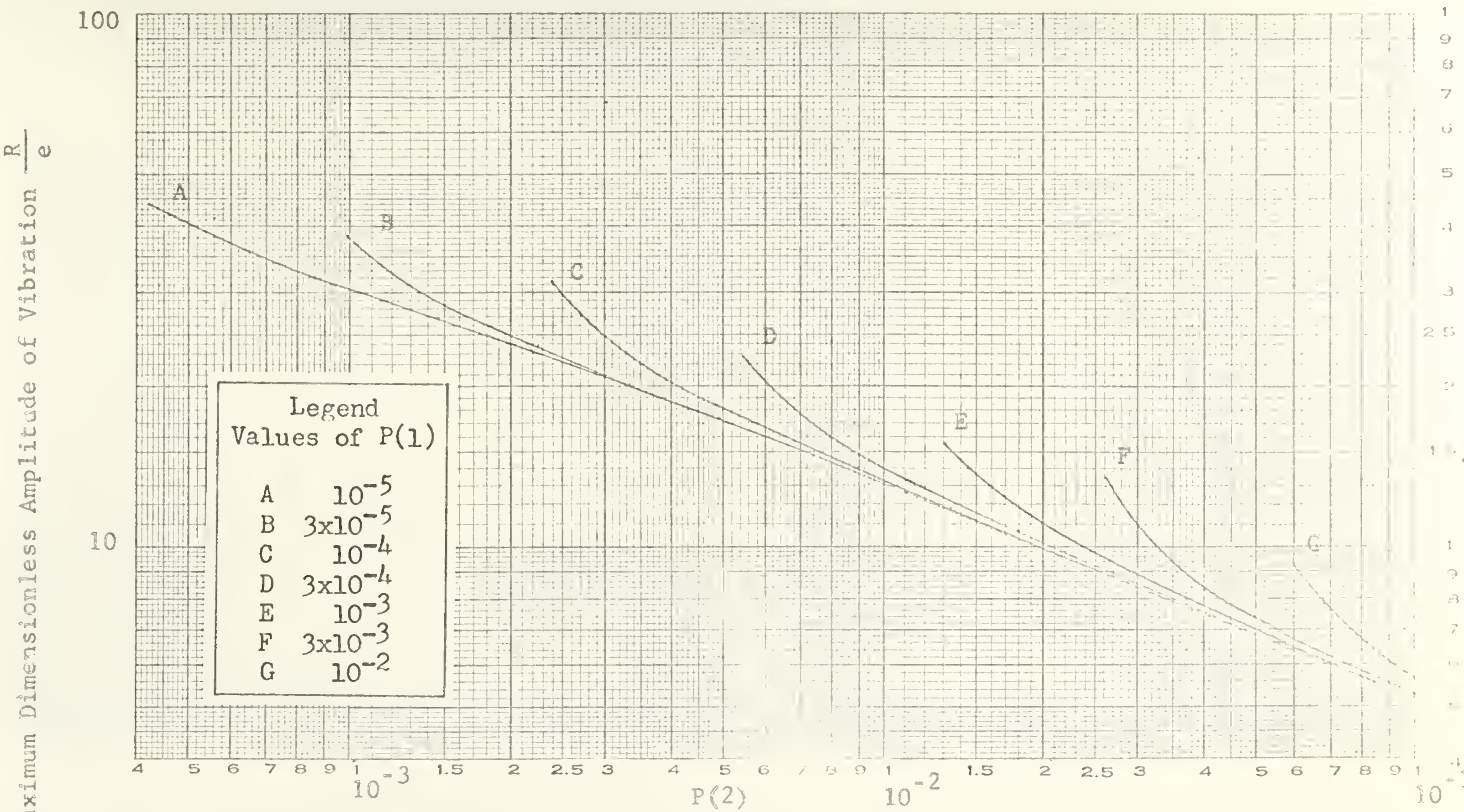


Fig. 17 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), P(3) = 1.0 (unity stiffness ratio). 1.00 per cent critical damping added (P(4) = 0.02).

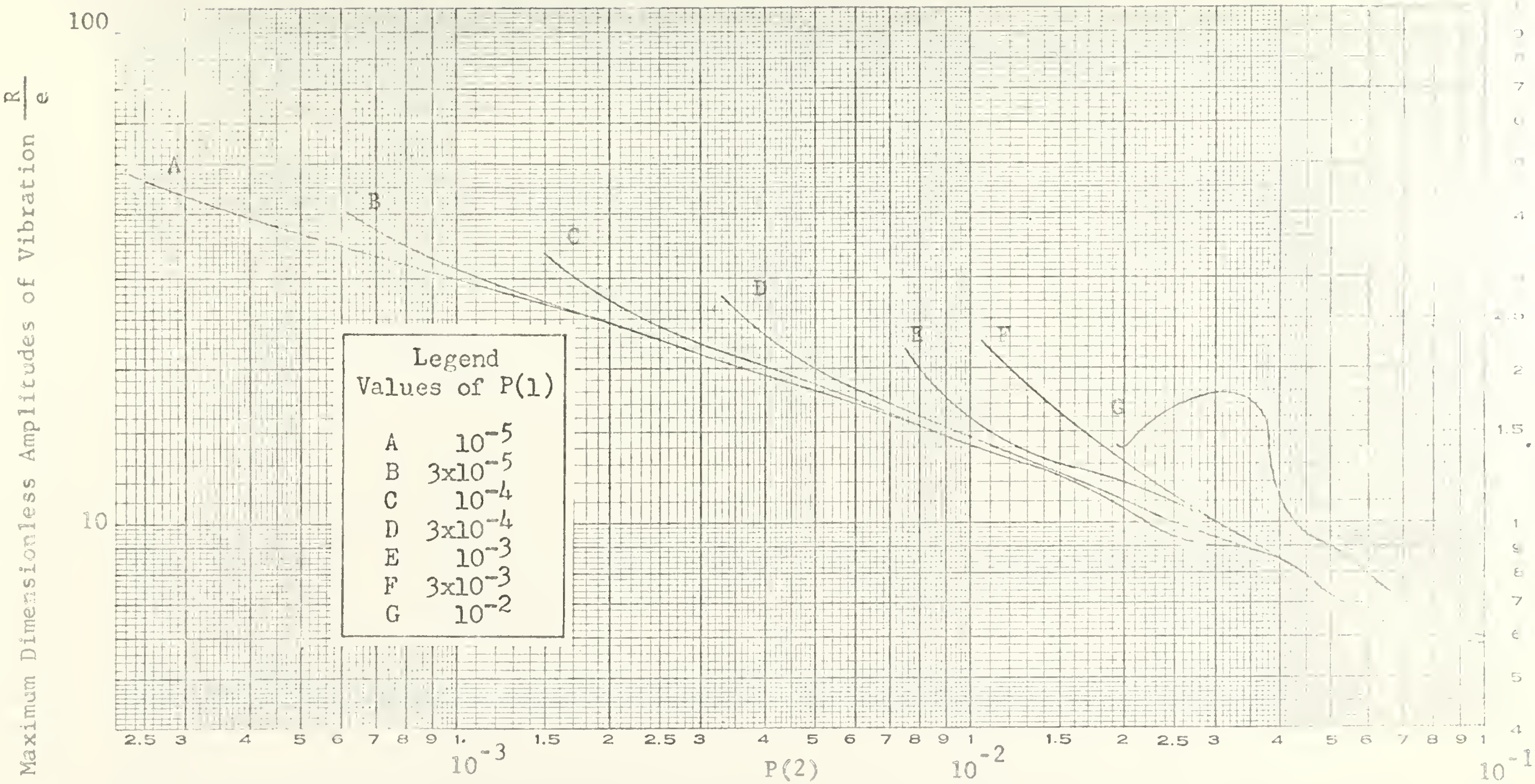


Fig. 18 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), stiffness ratio P(3) = 0.50, 1.00 per cent critical damping added (P(4) = 0.02).

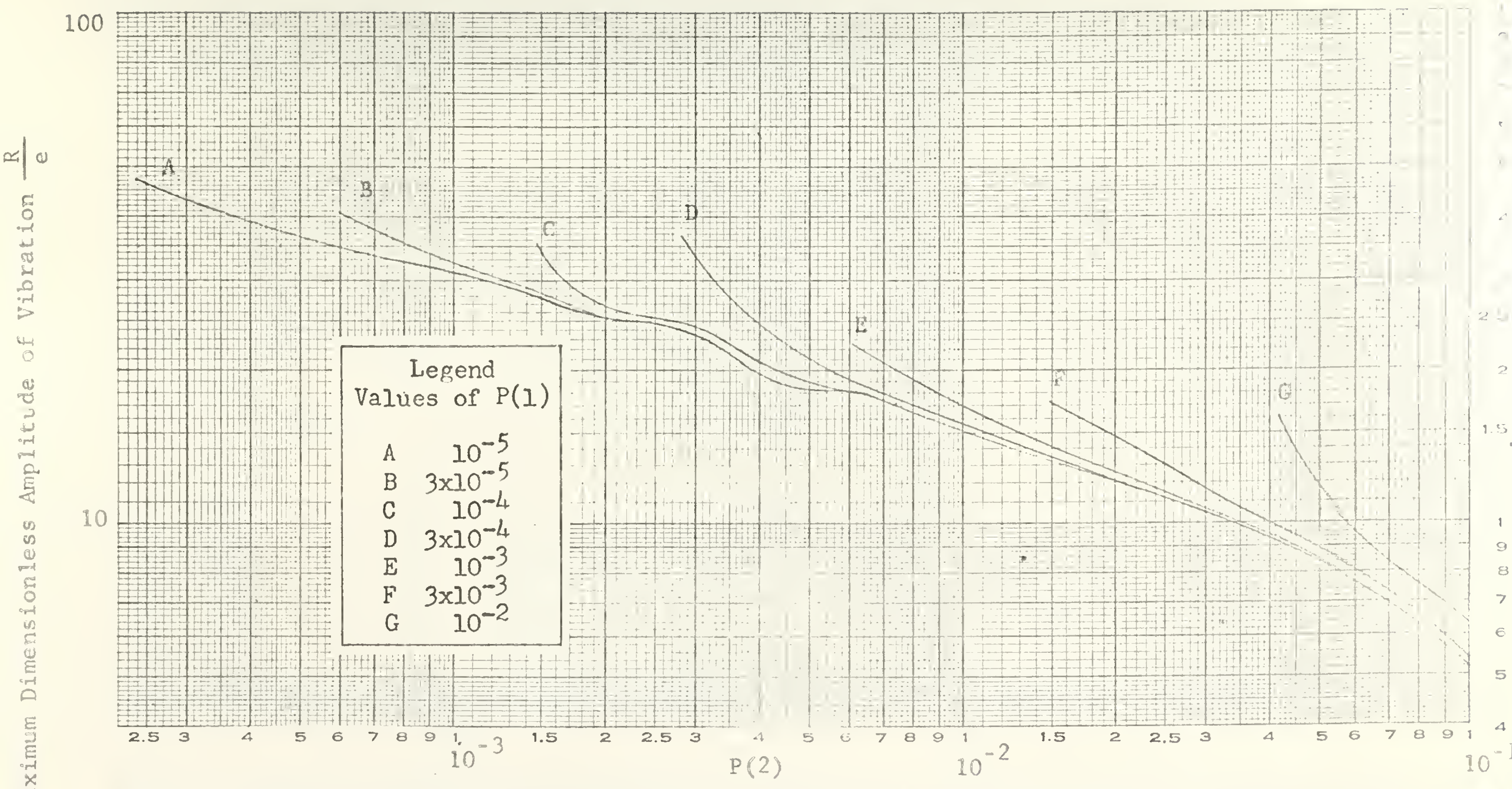


Fig. 19 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), stiffness ratio P(3) = 0.75, 1.00 per cent of critical damping added (P(4) = 0.02).

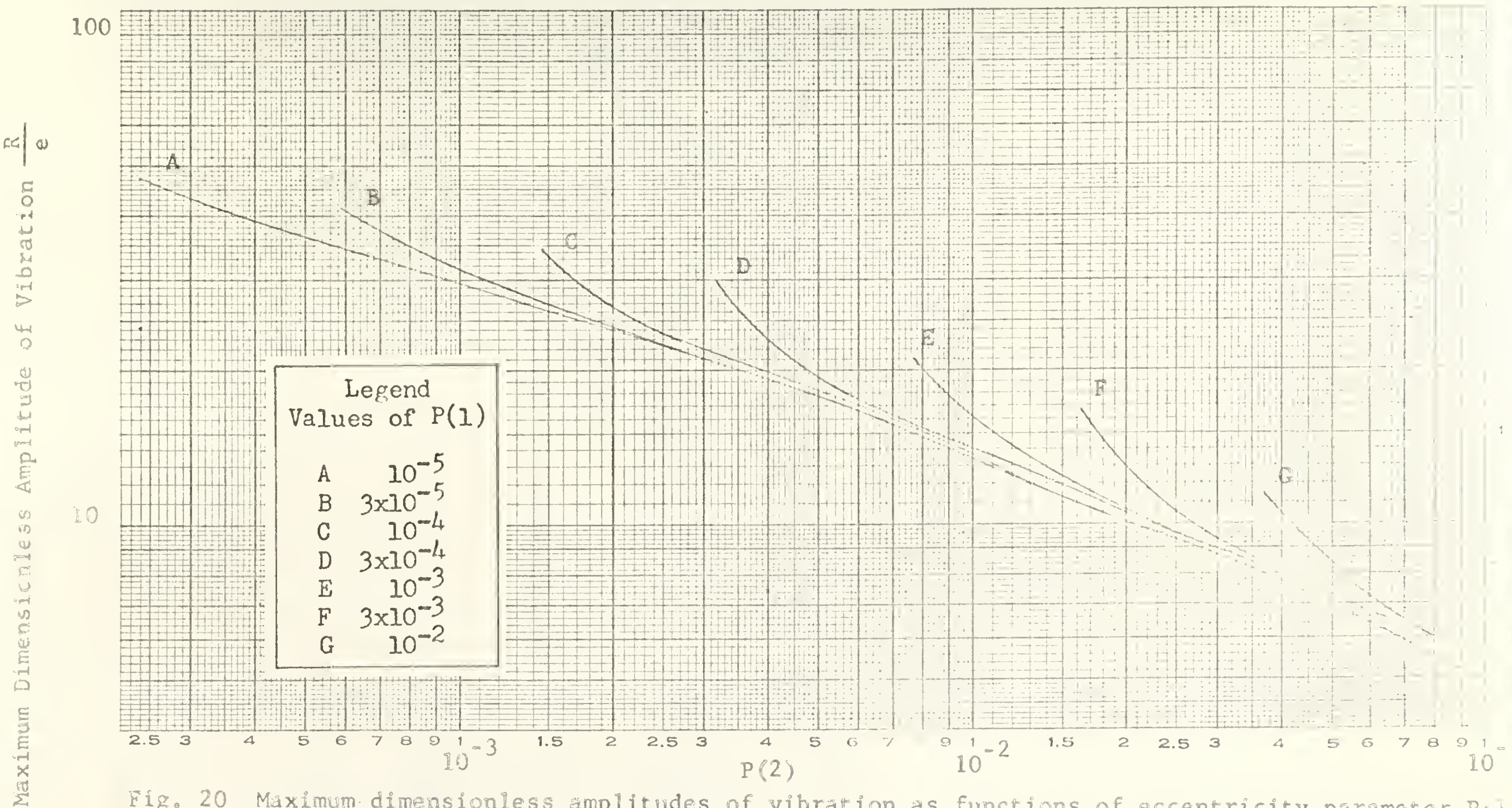


Fig. 20 Maximum dimensionless amplitudes of vibration as functions of eccentricity parameter P(1) and torque parameter P(2), stiffness ratio P(3) = 0.25, 1.00 per cent of critical damping added (P(4) = 0.02).

APPENDIX I

CONVERSION OF EQUATIONS FOR NUMERICAL INTEGRATION

The system of first order equations to be rearranged is restated for convenience.

$$\dot{\theta} = \alpha \tag{16}$$

$$\dot{\eta} = \beta \tag{17}$$

$$\dot{\zeta} = \gamma \tag{18}$$

$$\dot{\alpha} = P(2)P(5) + P(1)\left[\dot{\gamma}\sin\theta - \dot{\beta}\cos\theta\right] \tag{19}$$

$$\dot{\gamma} + P(4)\gamma + \zeta = \alpha^2\cos\theta + \dot{\alpha}\sin\theta \tag{20}$$

$$\dot{\beta} + P(4)P(6)\beta + P(3)\eta = \alpha^2\sin\theta - \dot{\alpha}\cos\theta \tag{21}$$

Equations 16, 17 and 18 have the required form:

$$\frac{dy_i}{dx} = f\left(x, y_1(x), y_2(x), \cdots, y_n(x)\right)$$

It remains, then, to reduce Eqs. 19, 20 and 21 to this same form.

Rearranging these equations:

$$\dot{\alpha} + P(1)\cos\theta\,\dot{\beta} - P(1)\sin\theta\,\dot{\gamma} = A$$

$$-\sin\theta\,\dot{\alpha} + \dot{\gamma} = B$$

$$\cos\theta\,\dot{\alpha} + \dot{\beta} = C$$

where:

$$A = P(2)P(5)$$

$$B = \alpha^2\cos\theta - P(4)\gamma - \zeta$$

$$C = \alpha^2\sin\theta - P(3)\eta - P(4)P(6)\beta$$

Employing Cramer's Rule.

$$\dot{\alpha} = \frac{\begin{vmatrix} A & P(1)\cos\theta & -P(1)\sin\theta \\ B & 0 & 1 \\ C & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & P(1)\cos\theta & -P(1)\sin\theta \\ -\sin\theta & 0 & 1 \\ \cos\theta & 1 & 0 \end{vmatrix}}$$

Thus:

$$\dot{\alpha} = \frac{C\,P(1)\cos\theta - B\,P(1)\sin\theta - A}{P(1)\cos^2\theta + P(1)\sin^2\theta - 1}$$

$$= \Big\{\left[\alpha^2\sin\theta - P(3)\eta - P(4)P(6)\beta\right]P(1)\cos\theta -$$

$$\left[\alpha^2\cos\theta - P(4)\gamma - \delta\right]P(1)\sin\theta - P(2)P(5)\Big\}\Big/\left[P(1)-1\right]$$

or:

$$\dot{\alpha} = \Big[P(1)\alpha^2\sin\theta\cos\theta - P(1)P(3)\eta\cos\theta -$$

$$P(1)P(4)P(6)\beta\cos\theta - P(1)\alpha^2\sin\theta\cos\theta +$$

$$P(1)P(4)\gamma\sin\theta + P(1)\delta\sin\theta -$$

$$P(2)P(5)\Big]\Big/\left[P(1)-1\right]$$

Finally:

$$\dot{\alpha} = \Big[P(1)\,P(3)\,\eta \cos\theta + P(1)\,P(4)\,P(6)\,\beta \cos\theta - P(1)\,P(4)\,\gamma \sin\theta - P(1)\,\xi \sin\theta + P(2)\,P(5) \Big] \Big/ \big[1 - P(1)\big] \tag{22}$$

$$\dot{\beta} = \frac{\begin{vmatrix} 1 & A & -P(1)\sin\theta \\ -\sin\theta & B & 1 \\ \cos\theta & C & 0 \end{vmatrix}}{\big[P(1) - 1\big]}$$

Thus:

$$\dot{\beta} = \Big[A\cos\theta + C\,P(1)\sin^2\theta + B\,P(1)\sin\theta\cos\theta - C \Big] \Big/ \big[P(1) - 1\big]$$

or:

$$\dot{\beta} = \Big\{ P(2)\,P(5)\cos\theta + \big[\alpha^2 \sin\theta - P(3)\,\eta - P(4)\,P(6)\,\beta\big] \cdot P(1)\sin^2\theta + \big[\alpha^2\cos\theta - P(4)\,\gamma - \xi\big] P(1)\sin\theta\cos\theta - \alpha^2\sin\theta + P(3)\,\eta + P(4)\,P(6)\,\beta \Big\} \Big/ \big[P(1) - 1\big]$$

Finally:

$$\dot{\beta} = \Big[-P(2)\,P(5)\cos\theta - P(1)\,\alpha^2\sin^3\theta + P(1)\,P(3)\,\eta\sin^2\theta + P(1)\,P(4)\,P(6)\,\beta\sin^2\theta - P(1)\,\alpha^2\sin\theta\cos^2\theta + P(1)\,P(4)\,\gamma\sin\theta\cos\theta + P(1)\,\xi\sin\theta\cos\theta + \tag{23}$$

$$\alpha^2 \sin\theta - P(3)\eta - P(4)P(6)\beta\Big] \Big/ \Big[1 - P(1)\Big]$$

$$\dot{\gamma} = \frac{\begin{vmatrix} 1 & P(1)\cos\theta & A \\ -\sin\theta & 0 & B \\ \cos\theta & 1 & C \end{vmatrix}}{[P(1)-1]}$$

Thus:

$$\dot{\gamma} = \Big[B P(1)\cos^2\theta - A\sin\theta - B + C P(1)\sin\theta\cos\theta\Big] \Big/ \Big[P(1)-1\Big]$$

or:

$$\dot{\gamma} = \Big\{\Big[\alpha^2\cos\theta - P(4)\gamma - \zeta\Big]P(1)\cos^2\theta - P(2)P(5)\sin\theta - \alpha^2\cos\theta + P(4)\gamma + \zeta + \Big[\alpha^2\sin\theta - P(3)\eta - P(4)P(6)\beta\Big]P(1)\sin\theta\cos\theta\Big\} \Big/ \Big[P(1)-1\Big]$$

Finally:

$$\dot{\gamma} = \Big[-P(1)\alpha^2\cos^3\theta + P(1)P(4)\gamma\cos^2\theta + P(1)\zeta\cos^2\theta + P(2)P(5)\sin\theta + \alpha^2\cos\theta - P(4)\gamma - \zeta - P(1)\alpha^2 \cdot \sin^2\theta\cos\theta + P(1)P(3)\eta\sin\theta\cos\theta + P(1)P(4)P(6)\beta\sin\theta\cos\theta\Big] \Big/ \Big[1 - P(1)\Big] \qquad (24)$$

APPENDIX II

SELECTION OF TIME INCREMENT

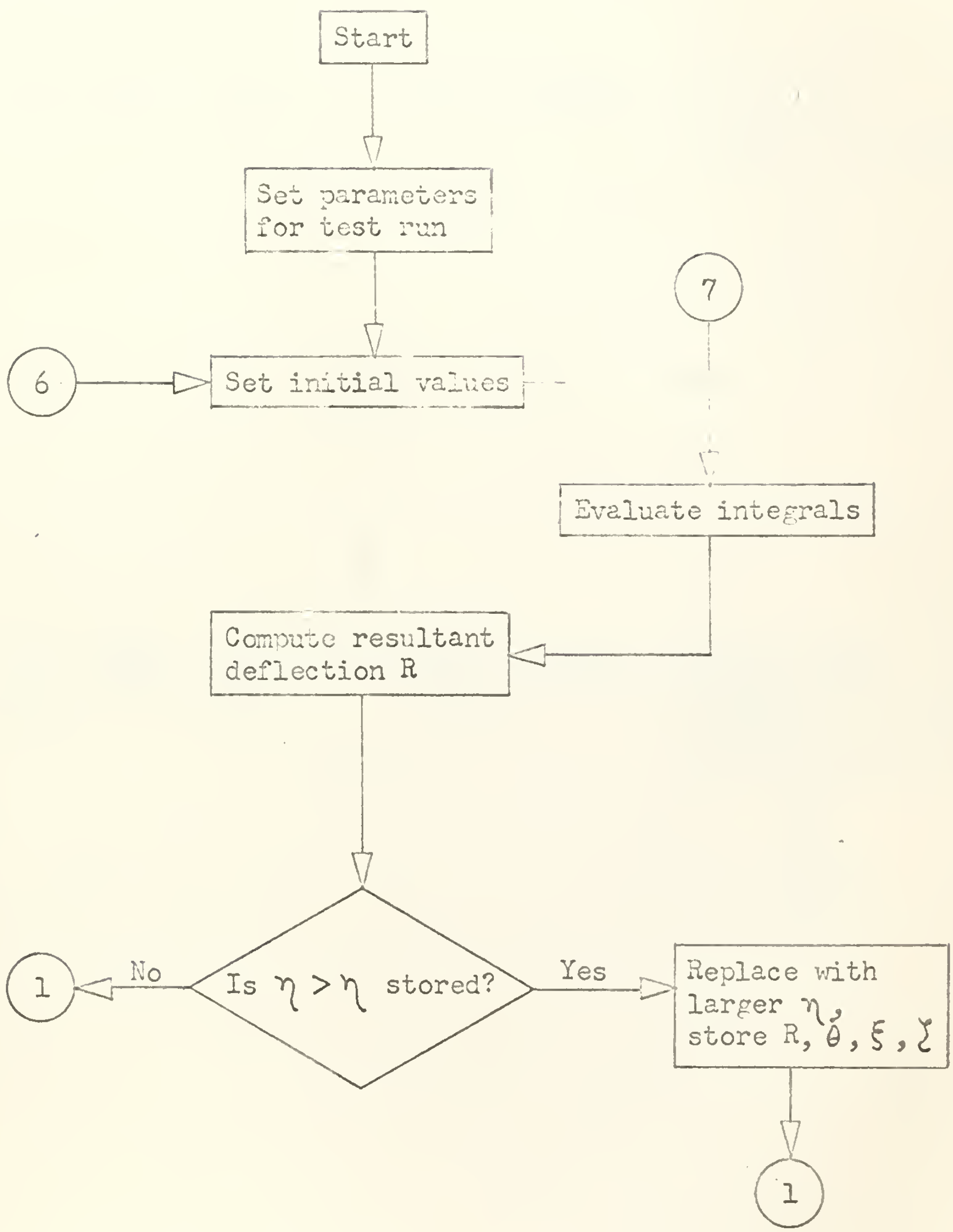
The selection of the time increment h was made by selecting several representative sets of parameters and obtaining the corresponding solutions for various h. This afforded two comparisons, one of the energy balances and another of the resultant deflections. Then the torque parameter was varied until the boundary between the regions of successful and unsuccessful accelerations was defined. This afforded a third check. Representative results for one set of parameters are presented in the accompanying table. The time increment of 0.1 was selected as the reference since smaller time increments produced no change in any of the three comparisons made for each set of parameters.

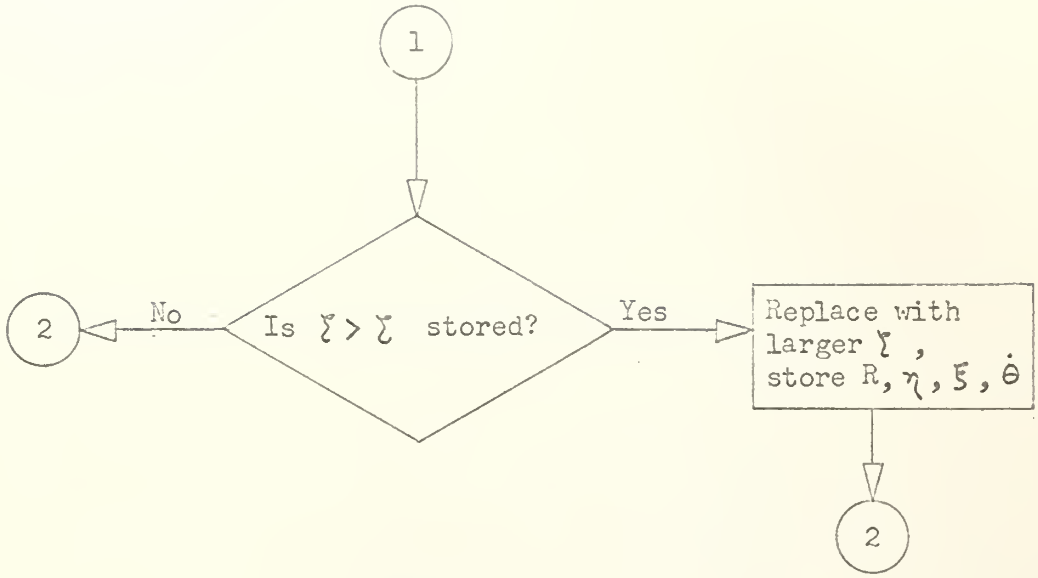
	Per cent difference as compared with results for h = 0.1		
h	Resultant	Energy	Boundary
0.1	0.00	0.00	0.00
0.2	0.00	0.00	5.00
0.3	0.61	0.00	0.00
0.4	2.23	2.40	0.00
0.5	1.82	7.20	0.00

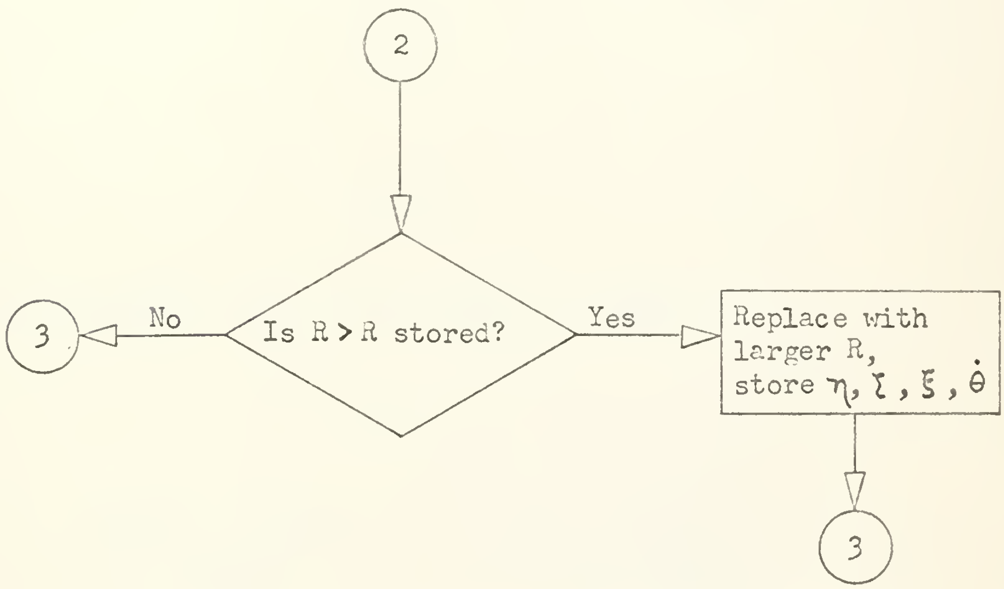
Effect of time increment h on resultant deflection, energy balance and location of the boundary between regions of successful and unsuccessful accelerations.

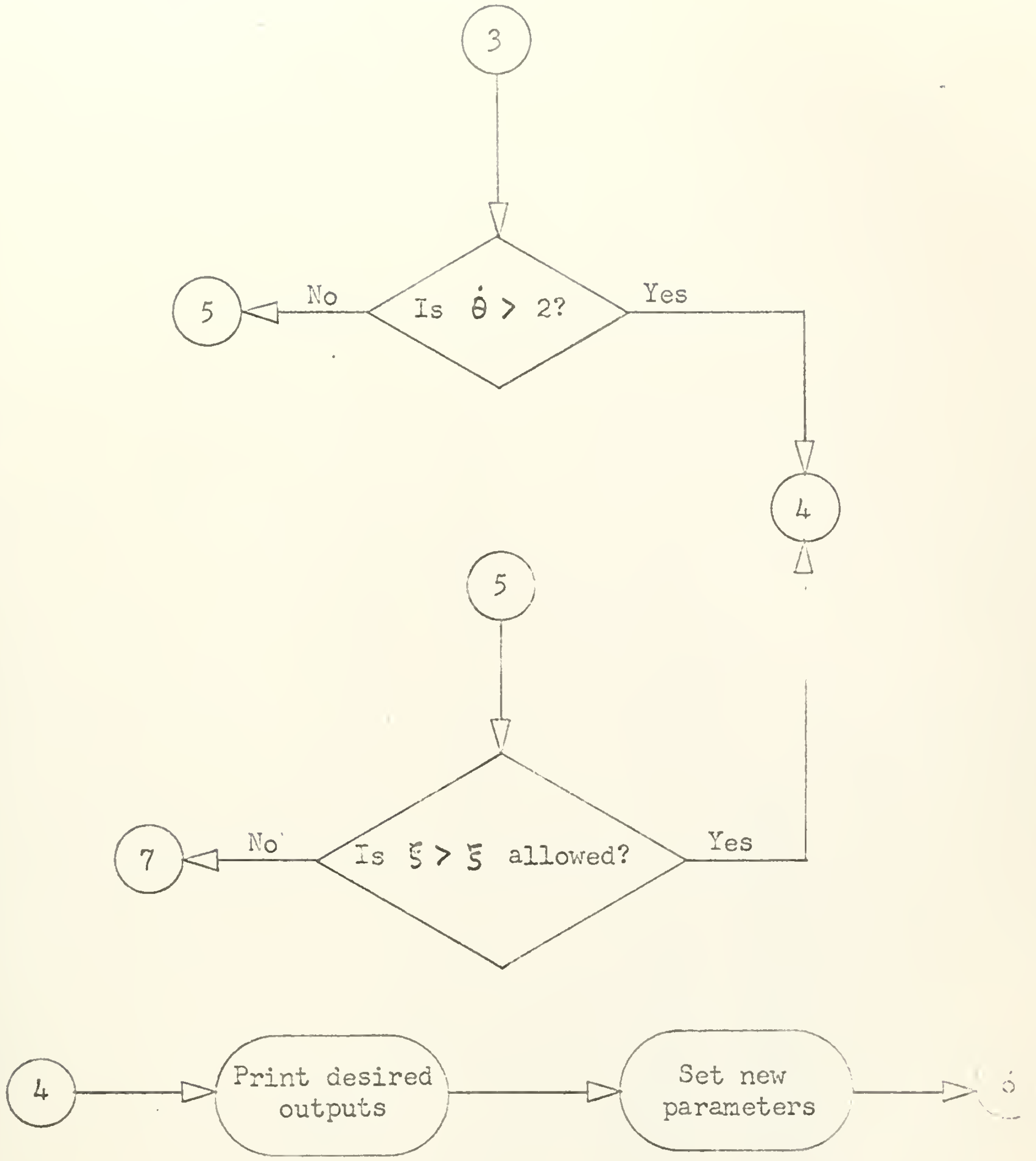
APPENDIX III

BLOCK DIAGRAM




1
Is ζ > ζ stored?
No
2
Yes
Replace with
larger ζ ,
store R, η, ξ, θ̇
2


2
Is R > R stored?
No
3
Yes
Replace with
larger R,
store η, ζ, ξ, θ̇
3


3
Is $\dot{\theta} > 2$?
No
Yes
5
4
5
Is $\xi > \xi$ allowed?
No
Yes
7
4
Print desired outputs
Set new parameters
6

```
      PROGRAM B2
      DIMENSION Y(6), D(6), P(6), YE(5), YZ(5), YR(5)
      COMMON P
      RUN=0.
      Z=0.
      P(1)=.01
      P(2)=.07
      P(3)=1.
      P(4)=0.
      R=0.
      DELP=0.
      H=.1
50    RUN=RUN+1.
      P(5)=1.
      P(6)=SQRTF(P(3))
      SPDMX=0.
      XMSPD=0.
      Y(1)=0.
      Y(2)=0.
      Y(3)=0.
      Y(4)=0.
      Y(5)=0.
      Y(6)=0.
      X=0.
      YE(1)=0.
      YZ(2)=0.
      YR(3)=0.
10    CALL RKUTTA(6,X,Y,H)
      X=X+H
      R=SQRTF(Y(1)*Y(1)+Y(2)*Y(2))
      IF(SPDMX-Y(6))75,20,20
75    SPDMX=Y(6)
      XMSPD=X
20    REV=Y(3)/6.28
21    IF(ABSF(Y(1))-ABSF(YE(1)))1,1,4
1     IF(ABSF(Y(2))-ABSF(YZ(2)))2,2,5
2     IF(ABSF(R)-ABSF(YR(3)))3,3,6
4     YE(1)=Y(1)
      YE(2)=Y(2)
      YE(3)=R
      YE(4)=Y(6)
      YE(5)=X
      GO TO 1
5     YZ(1)=Y(1)
```

```
      YZ(2)=Y(2)
      YZ(3)=R
      YZ(4)=Y(6)
      YZ(5)=X
      GO TO 2
6     YR(1)=Y(1)
      YR(2)=Y(2)
      YR(3)=R
      YR(4)=Y(6)
      YR(5)=X
3     IF (SENSE SWITCH 2) 91,92
91    PAUSE 2
      READ TAPE 2,(Y(I),I=1,6),X
      REWIND 2
92    IF(SENSE SWITCH 1) 22,7
7     IF(Y(6)-2.)10,8,8
22    PRINT 23
23    FORMAT(16HTIME CUTOFF USED)
      WRITE TAPE 2,(Y(I),I=1,6),X
8     PRINT 14,RUN,(P(I),I=1,5),(YE(I),YZ(I),YR(I),I=1,5)
14   0FORMAT(3HRUN2X,F4.0//1HP5X,1P5E14.2//3HY/E3X,1P3E14.2//3HX/E3X,
     11P3E14.2//4HRLNT2X,1P3E14.2//3HSPD3X,1P3E14.2//2HXI4X,1P3E14.2///)
      PRINT 80,SPDMX,XMSPD
80    FORMAT(6HMXSPD=2X,1PE14.2//6HXMSPD=2X,1PE14.2//)
      PRINT 40,(Y(I),I=1,3),(Y(6)),REV,    X,Y(5),Y(4)
40   0FORMAT(12HFINAL VALUES//4HY/E=2X,1PE14.2//4HX/E=2X,1PE14.2//
     16HTHETA=2X,1PE14.2//4HSPD=2X,1PE14.2//4HREV=2X,1PE14.2//
     2                    2HX=2X,1PE14.2//4HVEL=2X,1P2E14.2//)
      EGYIN=2.*Y(3)*P(2)/P(1)
      EGYSUM=Y(6)**2/P(1)+Y(2)**2+P(3)*Y(1)**2+Y(5)**2+Y(4)**2
     1-2.*Y(6)*(Y(5)*SINF(Y(3))-Y(4)*COSF(Y(3)))
      ERROR=(EGYIN-EGYSUM)*100./EGYIN
      PRINT 31,EGYIN,EGYSUM,ERROR
31   0FORMAT(6HEGYIN=3X,1PE14.2//7HEGYSUM=2X,1PE14.2//14HPERCENT ERROR=
     1 2X,1PE14.2////)
39    IF(SENSE SWITCH 1) 11,8
38    IF(DELP)11,11,12
11    PAUSE 1
      READ 60,P(1),P(2),DELP,Z,P(3),P(4)
60    FORMAT(6E8.2)
      GO TO 50
12    IF(Z-P(2))11,11,30
30    P(2)=DELP*P(2)
      GO TO 50
```

```
      END
      SUBROUTINE DERIV(D,Y,X)
      DIMENSION D(6),Y(6),P(6)
      COMMON P
      C1=SINF(Y(3))
      C2=COSF(Y(3))
      C3=Y(6)*Y(6)
      C4=C2*C2
      D(1)=Y(4)
      D(2)=Y(5)
      D(3)=Y(6)
     0D(4)=(-P(1)*C3       *C1          *C4                +P(1)*P(4)*Y(5)*C1
     1      *C2          +P(1)*Y(2)*C1          *C2          +C3       *C1
     2   -P(6)*P(4)*Y(4)-P(3)*Y(1)-P(2)*P(5)*C2              -P(1)*C3        *C1
     3*C1*C1        +P(1)*P(6)*P(4)*Y(4)*C1*C1              +P(1)*P(3)*Y(1)*
     4C1*C1            )*1./(1.-P(1))
     0D(5)=(C2            *C3        -P(4)*Y(5)-Y(2)-P(1)*C2           *C3    *
     1C1*C1            +P(1)*P(6)*P(4)*Y(4)*C1            *C2          +P(1)
     2*P(3)*Y(1)*C1           *C2          -P(1)*C2*C4              *C3
     3*P(1)*P(4)*Y(5)*C4                 +P(1)*Y(2)*C4                +P(2)*
     4P(5)*C1            )*1./(1.-P(1))
     0D(6)=(P(2)*P(5)+P(1)*P(6)*P(4)*Y(4)*C2            +P(1)*P(3)*Y(1)
     1*C2           -P(1)*P(4)*Y(5)*C1            -P(1)*Y(2)*C1           )*
     21./(1.-P(1))
      END
      SUBROUTINE RKUTTA(NUMBER,XVAR,YVARS,STEP)
      DIMENSION YVARS(30),AK(4,30),               DY(30),YC(30),C(3)
      C(1)=0.5
      C(2)=0.5
      C(3)=1.0
      CALL DE IV(DY,     YVARS,XVAR)
      DO 1 J=  NUMBER
    1 AK(1,J) STEP*DY(J)
      DO 2 I=2,4
      XC=XVAR +C(I-1)*STEP
      DO 3 J=1,NUMBER
    3 YC(J)= YVARS(J) + C(I-1)*AK(I-1,J)
      CALL DERIV(DY,YC,XC)
      DO 2 J=1,NUMBER
    2 AK(I,J)=STEP *DY(J)
      DO 4 J=1,NUMBER
    4 YVARS(J)=YVARS(J)+(AK(1,J)+2.*AK(2,J)+2.*AK(3,J)+AK(4,J))/6.0
      RETURN
      END
      END
```

APPENDIX [illegible]

TABULATED DATA

POINTS DEFINING THE BOUNDARY BETWEEN REGIONS OF SUCCESSFUL AND UNSUCCESSFUL ACCELERATIONS

	VALUES OF P(2) FOR P(3) = 1.00			
	P(4) = 0.00 (UNDAMPED)		P(4) = 0.02 (DAMPED)	
P(1)	UNSUCCESSFUL	SUCCESSFUL	UNSUCCESSFUL	SUCCESSFUL
1.00×10^{-2}	6.00×10^{-2}	6.30×10^{-2}	5.71×10^{-2}	6.00×10^{-2}
3.00×10^{-3}	2.62	2.74	2.49	2.62
1.00	1.26	1.31	1.26	1.31
3.00×10^{-4}	5.75×10^{-3}	6.00×10^{-3}	5.18×10^{-3}	5.44×10^{-3}
1.00	2.80	2.92	2.28	2.40
3.00×10^{-5}	1.24	1.30	9.42×10^{-4}	9.90×10^{-4}
1.00	6.00×10^{-4}	6.30×10^{-4}	4.03	4.24

	VALUES OF P(2) FOR P(3) = 0.25			
	P(4) = 0.00 (UNDAMPED)		P(4) = 0.02 (DAMPED)	
P(1)	UNSUCCESSFUL	SUCCESSFUL	UNSUCCESSFUL	SUCCESSFUL
1.00×10^{-2}	3.69×10^{-2}	3.88×10^{-2}	3.52×10^{-2}	3.69×10^{-2}
3.00×10^{-3}	1.63	1.71	1.55	1.63
1.00	8.00×10^{-3}	8.40×10^{-3}	7.25×10^{-3}	7.61×10^{-3}
3.00×10^{-4}	3.62	3.80	3.05	3.20
1.00	1.73	1.82	1.40	1.47
3.00×10^{-5}	7.80×10^{-4}	8.20×10^{-4}	5.60×10^{-4}	5.90×10^{-4}
1.00	3.78	3.95	2.27	2.38

Points Defining the Boundary (Cont.)

	VALUES OF P(2) FOR P(3) = 0.75			
	P(4) = 0.00 (UNDAMPED)		P(4) = 0.02 (DAMPED)	
P(1)	UNSUCCESSFUL	SUCCESSFUL	UNSUCCESSFUL	SUCCESSFUL
1.00×10^{-2}	4.17×10^{-2}	4.38×10^{-2}	3.97×10^{-2}	4.17×10^{-2}
8.00×10^{-3}	4.09	4.30	3.90	4.09
6.00	3.29	3.45	3.13	3.28
4.80	–	–	2.59	2.72
4.30	–	–	1.94	2.03
4.00	2.38	2.50	1.77	1.86
3.40	2.08	2.19	–	–
3.00	1.49	1.56	1.41×10^{-2}	1.49×10^{-2}
1.70	–	–	8.88×10^{-3}	9.33×10^{-3}
1.00	6.42×10^{-3}	6.75×10^{-3}	5.82	6.12
3.00×10^{-4}	2.76	2.89	2.70	2.83
1.00	1.27	1.33	1.40	1.47
3.00×10^{-5}	5.90×10^{-4}	6.20×10^{-4}	5.71×10^{-4}	6.00×10^{-4}
2.10	6.05	6.35	–	–
1.50	4.76	5.00	–	–
1.00	3.78	3.95	2.28×10^{-4}	2.40×10^{-4}

Points Defining the Boundary (Cont.)

	VALUES OF P(2) FOR P(3) = 0.50			
	P(4) = 0.00 (UNDAMPED)		P(4) = 0.02 (DAMPED)	
P(1)	UNSUCCESSFUL	SUCCESSFUL	UNSUCCESSFUL	SUCCESSFUL
1.00×10^{-2}	1.95×10^{-2}	2.05×10^{-2}	1.86×10^{-2}	1.95×10^{-2}
7.00×10^{-3}	1.54	1.62	2.67	2.80
5.00	1.19	1.25	1.98	2.08
3.80	9.71×10^{-3}	1.02	1.41	1.49
3.00	1.11×10^{-2}	1.17	1.01	1.06
2.60	1.43	1.50	1.27	1.33
2.20	1.24	1.30	–	–
1.80	1.05	1.10	1.05×10^{-2}	1.10×10^{-2}
1.50	9.04×10^{-3}	9.49×10^{-3}	–	–
1.20	8.57	9.00	–	–
1.00	7.54	7.90	7.16×10^{-3}	7.52×10^{-3}
3.00×10^{-4}	3.61	3.80	3.14	3.30
1.00	1.72	1.81	1.42	1.49
3.00×10^{-5}	7.61×10^{-4}	8.00×10^{-4}	5.71×10^{-4}	6.00×10^{-4}
1.00	3.72	3.90	2.22	2.33
2.80×10^{-3}	–	–	9.06×10^{-3}	9.52×10^{-3}
2.30	–	–	1.27×10^{-2}	1.33×10^{-2}

APPENDIX V

MAXIMUM AMPLITUDES OF VIBRATION FOR SUCCESSFUL ACCELERATIONS THROUGH THE CRITICAL SPEED REGIONS

$P(3) = 1.00 \quad P(4) = 0.00$		
$P(1)$	$P(2)$	R/e
1.00×10^{-2}	6.30×10^{-2}	9.59
	9.45	6.28
3.00×10^{-3}	2.74	15.8
	4.11	9.10
	6.16	7.07
	9.25	5.78
1.00×10^{-3}	1.31×10^{-2}	23.7
	1.96	12.8
	2.95	9.93
	4.42	8.00
	6.63	6.60
	9.95	5.54
3.00×10^{-4}	6.00×10^{-3}	30.9
	9.00	18.5
	1.35×10^{-2}	14.2
	2.02	11.4
	3.04	9.35
	4.56	7.76
	6.83	6.43
1.00×10^{-4}	2.92×10^{-3}	42.4
	4.38	26.1
	6.57	20.0
	9.85	16.0
	1.48×10^{-2}	13.0
	2.22	10.7
	3.33	8.86
	4.99	7.38
	7.48	6.13

Maximum Amplitudes of Vibration (Cont.)

P(3) = 1.00 P(4) = 0.00		
P(1)	P(2)	R/e
3.00×10^{-5}	1.30×10^{-3}	64.4
	1.95	38.8
	2.92	29.7
	4.39	23.6
	6.58	19.2
	9.87	15.7
	1.48×10^{-2}	12.9
	2.22	10.7
	3.33	8.83
	5.00	7.37
	7.50	6.12
1.00×10^{-5}	6.30×10^{-4}	90.1
	9.45	55.2
	1.42×10^{-3}	42.2
	2.13	33.6
	3.19	27.2
	4.79	22.2
	7.19	18.2
	1.08×10^{-2}	14.9
	1.62	12.3
	2.43	10.2
	3.64	8.48
	5.46	7.12
	8.19	5.96

Maximum Amplitudes of Vibration (Cont.)

$P(3) = 1.00$	$P(4) = 0.02$	
$P(1)$	$P(2)$	R/e
1.00×10^{-2}	6.00×10^{-2}	9.31
	9.00	6.04
3.00×10^{-3}	2.62	13.4
	4.00	8.38
	6.00	6.70
	9.00	5.56
1.00×10^{-3}	1.31	15.6
	2.00	11.0
	3.00	8.87
	4.50	7.34
	6.75	6.15
3.00×10^{-4}	5.44×10^{-3}	22.7
	8.00	16.0
	1.20×10^{-2}	12.9
	1.80	10.7
	2.70	8.94
	4.05	7.54
	6.07	6.34
	9.11	5.38
1.00×10^{-4}	2.40×10^{-3}	31.4
	4.00	20.4
	6.00	16.8
	9.00	14.0
	1.35×10^{-2}	11.8
	2.02	9.97
	3.04	8.43
	4.56	7.16
	6.83	6.03

Maximum Amplitudes of Vibration (Cont.)

P(3) = 1.00	P(4) = 0.02	
P(1)	P(2)	R/e
3.00×10^{-5}	9.90×10^{-4}	38.3
	1.50×10^{-3}	28.6
	2.25	23.9
	3.37	20.3
	5.06	17.3
	7.59	14.7
	1.14×10^{-2}	12.5
	1.71	10.6
	2.56	9.01
	3.84	7.62
	5.77	6.46
	8.65	5.51
1.00×10^{-5}	4.24×10^{-4}	44.0
	6.50	35.6
	9.75	30.8
	1.46×10^{-3}	26.8
	2.19	23.3
	3.29	20.1
	4.94	17.3
	7.40	14.8
	1.11×10^{-2}	12.6
	1.67	10.7
	2.50	9.08
	3.75	7.68
	5.62	6.54
	8.43	5.57

Maximum Amplitudes of Vibration (Cont.)

P(3) = 0.50	P(4) = 0.00	
P(1)	P(2)	R/e
1.00×10^{-2}	2.05×10^{-2}	22.7
	3.08	18.5
	4.62	10.6
	6.93	7.76
3.00×10^{-3}	1.17×10^{-2}	36.2
	1.75	17.9
	2.63	12.1
	3.95	9.97
	5.92	7.66
	8.88	6.34
1.00×10^{-3}	7.90×10^{-3}	75.0
	1.18×10^{-2}	18.2
	1.77	14.3
	2.65	11.0
	3.98	9.29
	5.97	7.40
	8.96	6.32
3.00×10^{-4}	3.80×10^{-3}	42.6
	5.69	25.6
	8.53	19.0
	1.28×10^{-2}	15.6
	1.92	12.6
	2.88	10.6
	4.32	8.86
	6.48	7.05
	9.72	6.21
1.00×10^{-4}	1.81×10^{-3}	62.0
	2.50	39.6
	3.75	30.6

$P(3) = 0.50$ $P(4) = 0.00$		
$P(1)$	$P(2)$	R/e
1.00×10^{-4}	5.63×10^{-3}	24.1
	8.44	18.4
	1.27×10^{-2}	15.4
	1.90	12.6
	2.85	10.6
	4.27	8.84
	6.41	7.06
	9.61	6.23
3.00×10^{-5}	8.00×10^{-4}	94.3
	1.20×10^{-3}	56.1
	1.80	43.2
	2.70	34.5
	4.05	28.0
	6.07	22.7
	9.11	18.4
	1.37×10^{-2}	15.2
	2.05	12.2
	3.08	10.4
	4.61	8.62
	6.92	6.87
1.00×10^{-5}	3.90×10^{-4}	131.
	6.00	77.9
	9.00	60.1
	1.35×10^{-3}	47.2
	2.02	39.1
	3.04	31.3
	4.56	24.8
	6.83	20.8
	1.03×10^{-2}	17.4

Maximum Amplitudes of Vibration (Cont.)

P(3) = 0.50	P(4) = 0.00	
P(1)	P(2)	R/e
1.00×10^{-5}	1.54×10^{-2}	14.0
	2.31	11.3
	3.46	9.83
	5.19	7.62
	7.78	6.41

P(3) = 0.50	P(4) = 0.02	
P(1)	P(2)	R/e
1.00×10^{-2}	1.95×10^{-2}	14.1
	3.00	16.8
	4.50	9.50
	6.75	7.26
3.00×10^{-3}	1.06×10^{-2}	22.6
	1.60	15.4
	2.40	11.3
	3.60	9.03
	5.40	7.19
	8.10	6.09
1.00×10^{-3}	7.52×10^{-3}	21.8
	1.13×10^{-2}	14.7
	1.69	12.6
	2.54	9.91
	3.81	8.73
	5.72	6.95
	8.58	6.08
3.00×10^{-4}	3.30×10^{-3}	27.6
	5.00	20.0
	7.50	16.6

Maximum Amplitudes of Vibration (Cont.)

$P(3) = 0.50$ $P(4) = 0.02$		
$P(1)$	$P(2)$	R/e
3.00×10^{-4}	1.13×10^{-2}	13.9
	1.69	11.9
	2.53	9.86
	3.80	8.58
	5.70	6.90
	8.54	6.08
1.00×10^{-4}	1.49×10^{-3}	33.8
	2.24	25.6
	3.36	21.5
	5.04	18.6
	7.56	15.3
	1.13×10^{-2}	13.7
	1.70	11.4
	2.55	9.80
	3.83	8.46
	5.74	6.85
	8.61	6.07
3.00×10^{-5}	6.00×10^{-4}	40.9
	9.00	32.9
	1.35×10^{-3}	28.1
	2.02	24.7
	3.04	21.3
	4.56	18.6
	6.83	16.4
	1.03×10^{-2}	13.9
	1.54	12.2
	2.31	9.68
	3.46	8.90
	5.19	7.05
	7.78	6.09

$P(3) = 0.50$ $P(4) = 0.02$		
P(1)	P(2)	R/e
1.00×10^{-5}	2.33×10^{-4}	47.9
	2.70	44.6
	4.05	39.1
	6.00	34.8
	9.00	30.8
	1.35×10^{-3}	27.3
	2.02	24.3
	3.04	21.2
	4.56	18.5
	6.83	16.3
	1.03×10^{-2}	13.9
	1.54	12.2
	2.31	9.67
	3.46	8.90
	5.19	7.05
	7.78	6.09

$P(3) = 0.50$ $P(4) = 0.02$		
$P(1)$	$P(2)$	R/e
1.00×10^{-5}	2.33×10^{-4}	47.9
	2.70	44.6
	4.05	39.1
	6.00	34.8
	9.00	30.8
	1.35×10^{-3}	27.3
	2.02	24.3
	3.04	21.2
	4.56	18.5
	6.83	16.3
	1.03×10^{-2}	13.9
	1.54	12.2
	2.31	9.67
	3.46	8.90
	5.19	7.05
	7.78	6.09

P(3) = 0.75	P(4) = 0.00	
P(1)	P(2)	R/e
1.00×10^{-2}	4.38×10^{-2}	20.0
	6.50	9.67
	9.75	7.22
3.00×10^{-3}	1.56×10^{-2}	23.2
	2.34	16.5
	3.51	11.9
	5.26	9.32
	7.90	7.36
1.00×10^{-3}	6.75×10^{-3}	29.2
	1.00×10^{-2}	24.4
	1.50	17.0
	2.25	13.7
	3.38	11.3
	5.06	9.08
	7.59	7.30
3.00×10^{-4}	2.89×10^{-3}	60.3
	4.35	33.9
	6.53	24.5
	9.79	20.5
	1.47×10^{-2}	15.6
	2.20	13.2
	3.30	11.1
	4.95	9.06
	7.43	7.28
1.00×10^{-4}	1.33×10^{-3}	162.
	2.00	49.3
	3.00	36.2
	4.50	28.1
	6.75	23.3

$P(3) = 0.75$ $P(4) = 0.00$		
$P(1)$	$P(2)$	R/e
1.00×10^{-4}	1.01×10^{-2}	19.0
	1.52	15.1
	2.28	12.9
	3.42	10.7
	5.13	8.73
	7.69	7.26
3.00×10^{-5}	6.20×10^{-4}	441.
	9.30	76.4
	1.39×10^{-3}	53.3
	2.09	41.8
	3.14	34.2
	4.71	26.0
	7.06	22.6
	1.06×10^{-2}	18.1
	1.59	14.6
	2.38	12.7
	3.58	10.5
	5.36	8.37
	8.04	7.20
1.00×10^{-5}	3.95×10^{-4}	138.
	4.30	115.
	6.50	79.4
	9.75	61.1
	1.46×10^{-3}	49.4
	2.19	40.2
	3.29	32.8
	4.94	25.3
	7.40	22.0
	1.11×10^{-2}	17.4

$P(3) = 0.75$ $P(4) = 0.00$		
$P(1)$	$P(2)$	R/e
1.00×10^{-5}	1.67×10^{-2}	14.5
	2.50	12.2
	3.75	10.4
	5.62	8.37
	8.43	7.05

$P(3) = 0.75$ $P(4) = 0.02$		
$P(1)$	$P(2)$	R/e
1.00×10^{-2}	4.17×10^{-2}	16.2
	6.30	9.11
	9.45	6.94
3.00×10^{-3}	1.49×10^{-2}	17.2
	2.24	13.9
	3.36	10.9
	5.04	8.87
	7.56	6.97
1.00×10^{-3}	6.12×10^{-3}	22.3
	9.20	18.8
	1.38×10^{-2}	14.6
	2.07	12.5
	3.10	10.6
	4.66	8.83
	6.99	6.92
3.00×10^{-4}	2.83×10^{-3}	36.5
	4.20	23.4
	6.30	18.8
	9.45	16.0
	1.42×10^{-2}	13.8

Maximum Amplitudes of vibration (Cont.)

P(3) = 0.75	P(4) = 0.02	
P(1)	P(2)	R/e
3.00×10^{-4}	2.13×10^{-2}	12.1
	3.19	10.3
	4.78	8.61
	7.18	6.72
1.00×10^{-4}	1.47×10^{-3}	35.2
	2.20	25.8
	3.30	23.5
	4.95	19.0
	7.42	17.3
	1.11×10^{-2}	14.5
	1.67	13.0
	2.51	11.1
	3.76	9.59
	5.64	7.87
	8.46	6.66
3.00×10^{-5}	6.00×10^{-4}	40.7
	9.00	33.8
	1.35×10^{-3}	29.4
	2.02	25.3
	3.04	23.5
	4.56	19.0
	6.83	17.4
	1.03×10^{-2}	14.7
	1.54	13.2
	2.31	11.6
	3.46	9.66
	5.19	7.92
	7.78	6.27

Maximum Amplitudes of Vibration (cont.)

[illegible]		
P(1)	P(2)	R/e
1.00 X10^-5	[illegible]	[illegible]
	[illegible]	[illegible]
	5.40	[illegible]
	8.10	[illegible]
	1.21 X10^-3	29.4
	1.82	25.7
	2.73	[illegible]
	4.10	19.1
	6.15	[illegible]
	9.23	[illegible]
	1.38 X10^-2	13.5
	2.08	12.0
	3.11	10.3
	4.67	8.67
	7.01	6.85

P(3) = 0.25 P(4) = 0.00		
P(1)	P(2)	R/e
1.00 X10^-2	3.88 X10^-2	12.7
	6.00	7.97
	9.00	6.23
3.00 X10^-3	1.71 X10^-2	21.5
	2.50	12.1
	3.75	9.05
	5.62	7.43
	8.44	6.21
1.00 X10^-3	8.30 X10^-3	26.7
	1.25 X10^-2	16.7

Maximum Amplitudes of Vibration (Cont.)

P(3) = 0.[illegible]	P(?) = [illegible]	
P(1)	[illegible]	[illegible]
1.00×10^{-3}	$1.87 \times 10^{-[illegible]}$	[illegible].7
	2.81	10.5
	4.22	8.47
	[illegible].33	6.67
	9.49	5.71
3.00×10^{-4}	3.70×10^{-3}	39.5
	5.70	24.5
	8.55	18.8
	1.28×10^{-2}	15.1
	1.92	12.4
	2.89	9.67
	4.33	8.08
	6.49	6.56
	9.74	5.52
1.00×10^{-4}	1.82×10^{-3}	57.2
	2.75	34.9
	4.12	26.7
	6.19	21.2
	9.28	16.8
	1.39×10^{-2}	13.8
	2.09	11.3
	3.13	9.70
	4.69	7.91
	7.04	6.13
3.00×10^{-5}	8.20×10^{-4}	82.3
	1.23×10^{-3}	51.6
	1.84	39.9
	2.77	31.0
	[illegible]	25.7

Maximum Amplitudes of Vibration (Cont.)

P(3) = [illegible] P(4) = [illegible]		
P(1)	P(2)	R/e
3.00×10^{-5}	6.23×10^{-3}	[illegible]
	9.34	16.4
	1.40×10^{-2}	13.7
	2.10	11.3
	3.15	9.67
	4.73	7.90
	7.09	6.15
1.00×10^{-5}	3.95×10^{-4}	118.
	6.00	[illegible]
	9.00	55.5
	1.35×10^{-3}	45.0
	2.02	36.4
	3.03	29.9
	4.54	[illegible]
	6.82	19.4
	1.02×10^{-2}	15.8
	1.53	13.6
	2.30	10.5
	3.45	8.85
	5.18	7.63
	7.77	[illegible]

P(3) = 0.25 P(4) = 0.02		
P(1)	P(2)	R/e
1.00×10^{-2}	3.69×10^{-2}	11.6
	5.50	7.74
	8.25	5.76
3.00×10^{-3}	1.63×10^{-2}	16.8

Maximum Amplitudes of Vibration (Cont.)

P(3) = 0.25 P(4) = 0.02		
P(1)	P(2)	R/e
3.00×10^{-3}	2.45×10^{-2}	10.7
	3.67	8.85
	5.51	7.05
	8.27	5.91
1.00×10^{-3}	7.61×10^{-3}	21.2
	1.14×10^{-2}	14.6
	1.71	11.5
	2.56	9.60
	3.85	8.29
	5.77	6.37
	8.66	5.78
3.00×10^{-4}	3.20×10^{-3}	29.9
	4.80	19.8
	7.20	16.2
	1.08×10^{-2}	13.6
	1.62	11.4
	2.43	9.40
	3.64	8.32
	5.47	6.71
	8.20	5.88
1.00×10^{-4}	1.47×10^{-3}	34.4
	2.20	25.5
	3.30	21.4
	4.95	18.3
	5.20	18.0
	7.80	15.4
	1.17×10^{-2}	12.7
	1.75	10.9
	2.63	9.46
	3.95	8.11

$P(3) = 0.25$	$P(4) = 0.02$	
$P(1)$	$P(2)$	R/e
1.00×10^{-4}	6.00×10^{-2}	6.33
	9.00	5.62
3.00×10^{-5}	5.90×10^{-4}	41.5
	8.85	33.1
	1.33×10^{-3}	28.2
	1.99	24.4
	3.00	21.1
	4.50	18.2
	6.75	15.7
	1.01×10^{-2}	13.5
	1.52	11.7
	2.28	9.52
	3.42	8.17
	5.13	7.13
	7.69	5.92
1.00×10^{-5}	2.38×10^{-4}	47.2
	3.60	40.5
	5.40	35.9
	8.10	31.8
	1.21×10^{-3}	28.0
	1.82	24.7
	2.73	21.8
	4.10	19.0
	6.15	16.4
	9.23	13.7
	1.38×10^{-2}	11.9
	2.08	10.0
	3.11	8.73
	4.67	7.37
	7.01	5.81